Coupling linearity and twist: an extension of the Poincaré–Birkhoff Theorem for Hamiltonian systems

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Abstract. We provide an extension of the Poincaré–Birkhoff Theorem for systems coupling linear components with twisting components. Applications are given both to weakly coupled Hamiltonian systems where, e.g., a superlinear or sublinear behaviour is assumed in order to recover the needed twist conditions, and to local perturbations of superintegrable systems, showing the survival of a number of periodic solutions from a lower-dimensional torus.

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1. Introduction and main result

We consider the Hamiltonian system

$$\dot{z} = J\nabla H(t,z)\,, \tag{HS}$$

and we assume the Hamiltonian function $H\colon \mathbb{R}\times\mathbb{R}^{2N}\to\mathbb{R}$ to be continuous, T-periodic in its first variable t, and continuously differentiable with respect to the variable z, with corresponding gradient $\nabla H(t,z)$.

Here $J = \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$ denotes the standard symplectic matrix. It will be often used in the sequel, also in spaces having a different dimension.

For $z\in\mathbb{R}^{2N}$, we use the notation $z=(x,y)$, with $x=(x_1,\dots,x_N)\in\mathbb{R}^N$ and $y=(y_1,\dots,y_N)\in\mathbb{R}^N$. Moreover, we gather into four groups the variables of x and y, respectively, thus writing

$$x=(x^a,x^b,x^c,x^d), \qquad\qquad y=(y^a,y^b,y^c,y^d),$$

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where, for some nonnegative integers N^a, N^b, N^c, N^d,

$$x^a = (x_1^a, \dots, x_{N^a}^a) \in \mathbb{R}^{N^a}, \qquad y^a = (y_1^a, \dots, y_{N^a}^a) \in \mathbb{R}^{N^a},$$
$$x^b = (x_1^b, \dots, x_{N^b}^b) \in \mathbb{R}^{N^b}, \qquad y^b = (y_1^b, \dots, y_{N^b}^b) \in \mathbb{R}^{N^b},$$
$$x^c = (x_1^c, \dots, x_{N^c}^c) \in \mathbb{R}^{N^c}, \qquad y^c = (y_1^c, \dots, y_{N^c}^c) \in \mathbb{R}^{N^c},$$
$$x^d = (x_1^d, \dots, x_{N^d}^d) \in \mathbb{R}^{N^d}, \qquad y^d = (y_1^d, \dots, y_{N^d}^d) \in \mathbb{R}^{N^d}.$$

We also introduce the notation

$$z^a = (x^a, y^a), \qquad z^b = (x^b, y^b), \qquad z^c = (x^c, y^c), \qquad z^d = (x^d, y^d).$$

Notice that one or more of these integers could be equal to zero, in which case the corresponding group will not be taken into account; for example, if $N^a = 0$, then x^a, y^a and z^a will disappear from the list.

We assume that

$H(t, x, y)$ is 2π-periodic in each of the variables included in x^a, x^b, y^a, y^c.

The total number of variables in which our Hamiltonian function is 2π-periodic is thus

$$M := N^a + N^b + N^a + N^c.$$

Under this setting, T-periodic solutions $z(t)$ of (HS) appear in equivalence classes made of those solutions whose components in $x^a(t), x^b(t), y^a(t), y^c(t)$ differ by an integer multiple of 2π. We say that two T-periodic solutions are *geometrically distinct* if they do not belong to the same equivalence class.

The existence of periodic solutions for this kind of systems has been studied by many authors, starting from the pioneering paper by Conley and Zehnder [6]. Among several others, we mention the abstract variational theorems of Chang [4] and Szulkin [30, 31], which are at the foundation of our approach.

We now assume that there exists a symmetric $N^d \times N^d$ matrix $\mathbb{A}(t)$, T-periodic and continuous in t, satisfying the nonresonance condition

$$z(t) \equiv 0 \quad \text{is the only } T\text{-periodic solution of } \dot z(t) = J\mathbb{A}(t) z(t)\,, \tag{1.1}$$

and such that the Hamiltonian function can be written as

$$H(t, z) = \tfrac{1}{2}\langle \mathbb{A}(t) z^d, z^d\rangle + K(t, z)\,,$$

where $K(t, z)$ has a bounded gradient with respect to z, i.e., there exists a constant $C_1 > 0$ for which

$$|\nabla K(t, z)| \le C_1\,, \qquad \text{for every } (t, z) \in \mathbb{R} \times \mathbb{R}^{2N}. \tag{1.2}$$

(We denote by $\langle \cdot, \cdot \rangle$ the Euclidean scalar product, and by $|\cdot|$ its associated norm.) Let us also introduce a $\mathcal{C}^1$-function $h\colon \mathbb{R}^{N^b+N^c} \to \mathbb{R}$ and a regular symmetric $(N^b + N^c) \times (N^b + N^c)$ matrix $\mathbb{S}$ such that, for some constant $C_2 > 0$,

$$\left|h(v) - \tfrac{1}{2}\langle \mathbb{S}v, v\rangle\right| \le C_2 \quad \text{and} \quad |\nabla h(v) - \mathbb{S}v| \le C_2\,, \quad \text{for every } v \in \mathbb{R}^{N^b+N^c}, \tag{1.3}$$

and let

$$D = \{v \in \mathbb{R}^{N^b+N^c} : \nabla h(v) = 0\}\,. \tag{1.4}$$

We assume that such a set is compact. Our main result is the following.

Theorem 1.1. *In the above setting, assume moreover that there exists* $\rho > 0$ *such that, for any solution* $z(t)$ *of* (HS) *with* $0 < \mathrm{dist}((y^b(0), x^c(0)), D) < \rho$, *one has*

$$(x^b(T) - x^b(0), y^c(T) - y^c(0)) \notin \{\lambda J \nabla h((y^b(0), x^c(0))) : \lambda \geq 0\}\,. \tag{1.5}$$

Then, system (HS) *has at least* $M + 1$ *geometrically distinct* T*-periodic solutions* $z(t)$*, with the property that*

$$(y^b(0), x^c(0)) \in D\,. \tag{1.6}$$

Moreover, if all the T*-periodic solutions of* (HS) *are non degenerate, then there are at least* 2^M *of them.*

It can be seen that, when $N^d = 0$ in the above statement, we can replace $0 < \mathrm{dist}((y^b(0), x^c(0)), D) < \rho$ by $(y^b(0), x^c(0)) \in \partial D$. In this case, the *avoiding cones condition* (1.5) has been extensively discussed in [13]. In particular, if $N^c = N^d = 0$ and D is a convex set, it has been shown in [13, Theorem 12] that the same conclusion of Theorem 1.1 is reached assuming, for any solution $z(t)$ of (HS) with $y^b(0) \in \partial D$, that

$$x^b(T) - x^b(0) \notin \mathcal{N}_D(y^b(0))\,,$$

where $\mathcal{N}_D(y^b)$ denotes the normal cone to D at $y^b \in \partial D$.

Theorem 1.1 is an extension of the main result in [18] (cf. also [13, 19]). It generalizes the results in [4] and [24], where a similar setting was proposed, assuming a coercivity condition in y^b and x^c instead of our avoiding cones condition. It also generalizes results from [14, 23, 26].

In the next section we provide some corollaries of Theorem 1.1 which will open the way to the applications we have in mind, and illustrate in more concrete terms the meaning of the *twist condition* (1.5). The proof of Theorem 1.1 is carried out in Section 3. In the last three sections we discuss some possible applications of our result.

In Section 4 we present a local existence result concerning the perturbation of a completely resonant lower dimensional torus, thus extending the result in [11] for the case $N^d = 0$. Under suitable nondegeneracy conditions, we prove the survival of periodic solutions under a small perturbation of the autonomous system.

In Section 5 we propose instead a global existence result, considering the weakly coupling of a linear system with systems which have a superlinear behaviour at infinity.

Finally, in Section 6, we survey other possible applications to weakly coupled systems, discussing how a sublinear or pendulum-like behaviour can be handled similarly as the superlinear one studied in the previous section.

2. Corollaries and remarks

Let D be a given convex body of $\mathbb{R}^{N^b+N^c}$ (i.e., a closed convex bounded set with a nonempty interior), and let $\pi_D\colon \mathbb{R}^{N^b+N^c} \to D$ denote the projection on it. When there is no ambiguity, we shorten $\pi_D v$ for $\pi_D(v)$. Moreover, when D has a smooth boundary, let $\mathcal{N}_D(\zeta)$ and $\nu_D(\zeta)$ be respectively the normal cone and the unit outward normal to the boundary of D at some point $\zeta \in \partial D$.

The aim of this section is to provide some conditions which guarantee the possibility of constructing a function h verifying (1.3) and (1.4), and for which the avoiding cones condition (1.5) holds. We refer to [13] for further details. Here is our first corollary.

Corollary 2.1. *Assume that D has a smooth boundary, and that there is a constant $\rho > 0$ such that for any solution $z(t)$ of* (HS) *with*

$$0 < \operatorname{dist}((y^b(0), x^c(0)), D) < \rho$$

one has

$$(x^b(T) - x^b(0), y^c(T) - y^c(0)) \notin \mathcal{N}_D(\pi_D((y^b(0), x^c(0)))). \tag{2.1}$$

Then, the conclusion of Theorem 1.1 *holds.*

Proof. We need to consider a $\mathcal{C}^\infty$-smooth function $\sigma\colon \mathbb{R} \to \mathbb{R}$ such that

$$\sigma(s) = \begin{cases} 0 & \text{if } s \le 0\,, \\ 1 & \text{if } s \ge 1\,, \end{cases} \qquad \sigma'(s) > 0 \text{ if } s \in \,]0,1[\,.$$

We define the function $h\colon \mathbb{R}^{N^b+N^c} \to \mathbb{R}$ by

$$h(v) = \xi(v)|v - \pi_D v|^2\,,$$

where

$$\xi(v) = \begin{cases} 0 & \text{if } v \in D\,, \\ \frac{1}{2}\sigma(|v - \pi_D v|) & \text{if } v \notin D\,. \end{cases}$$

Notice that

$$\nabla\xi(v) = \frac{\sigma'(|v - \pi_D v|)}{2|v - \pi_D v|}(v - \pi_D v)\,, \qquad \text{for every } v \notin D\,. \tag{2.2}$$

Then, if $v \notin D$,

$$\nabla h(v) = \left[\tfrac{1}{2}\sigma'(|v - \pi_D v|)|v - \pi_D v| + \sigma(|v - \pi_D v|)\right](v - \pi_D v)\,,$$

hence (1.3) and (1.4) hold, with $\mathbb{S} = \mathbb{I}$. Moreover, since $\nabla h(v)$ has the same direction as $\nu_D(\pi_D v)$, for every $v \notin D$, we see that (2.1) is equivalent to (1.5), hence the result follows from Theorem 1.1. □

Remark 2.2. Notice that assumption (2.1) can be replaced by

$$(x^b(T) - x^b(0), y^c(T) - y^c(0)) \notin -\mathcal{N}_D(\pi_D((y^b(0), x^c(0)))). \tag{2.3}$$

In the proof, it is sufficient to take $h(v) = -\xi(v)\,|v - \pi_D v|^2$, and the result follows in a similar way.

Here is our second corollary, where we assume that D is *strongly convex*, meaning that, for any $v \in \partial D$, the height function $\eta \mapsto \langle \eta - v, -\nu_D(v)\rangle$ has a nondegenerate minimum at $\eta = v$.

Corollary 2.3. *Assume that D is strongly convex, with a smooth boundary, and that there exist a symmetric regular $(N^b + N^c) \times (N^b + N^c)$ matrix $\mathbb{B}$ and a constant $\rho > 0$ such that, for any solution $z(t)$ of* (HS) *with* $0 < \operatorname{dist}((y^b(0), x^c(0)), D) < \rho$, *one has*

$$\langle (x^b(T) - x^b(0), y^c(T) - y^c(0)),\ \mathbb{B}\nu_D(\pi_D((y^b(0), x^c(0))))\rangle > 0\,. \tag{2.4}$$

Then, the conclusion of Theorem 1.1 *holds.*

Proof. We consider the $\mathcal{C}^\infty$-smooth function $\xi(v)$ introduced in the proof of Corollary 2.1, and define

$$h(v) = -\xi(v)\langle \mathbb{B}(v - \pi_D v), v - \pi_D v\rangle\,.$$

By the chain rule, if $v \notin D$,

$$\nabla h(v) = -\langle \mathbb{B}(v - \pi_D v), v - \pi_D v\rangle \nabla\xi(v) - 2\xi(v)(\mathrm{Id} - \pi_D'(v))^* \mathbb{B}(v - \pi_D v)\,.$$

For $|v|$ large enough, since $\xi(v) = \frac{1}{2}$ and $\nabla\xi(v) = 0$, we have

$$\begin{aligned} |\nabla h(v) + \mathbb{B}v| &= |\mathbb{B}\pi_D v + \pi_D'(v)^* \mathbb{B}(v - \pi_D v)| \\ &\le |\mathbb{B}\pi_D v| + \|\pi_D'(v)^*\|\, \|\mathbb{B}\|\, |v - \pi_D v|\,. \end{aligned}$$

Since D is strongly convex, by [18, Lemma 2.2] there is a constant $c > 0$ such that

$$\|\pi_D'(v)\|\, |v - \pi_D v| \le c\,, \qquad \text{for every } v \notin D\,,$$

hence (1.3) holds, with $\mathbb{S} = -\mathbb{B}$. Moreover, if $v \notin D$,

$$\begin{aligned} \langle \nabla h(v), -\mathbb{B}\nu_D(\pi_D v)\rangle =& \langle \mathbb{B}(v - \pi_D v), v - \pi_D v\rangle \langle \nabla\xi(v), \mathbb{B}\nu_D(\pi_D v)\rangle + \\ &+ 2\xi(v)\langle (\mathrm{Id} - \pi_D'(v))^* \mathbb{B}(v - \pi_D v), \mathbb{B}\nu_D(\pi_D v)\rangle\,. \end{aligned}$$

Now, in view of (2.2), $\nabla\xi(v)$ has the same direction as $v - \pi_D v$. Since $v - \pi_D v = \operatorname{dist}(v, \partial D)\nu(\pi_D v)$, the first term in the right hand side of the equality is nonnegative. On the other hand, by [18, Lemma 2.2], we have that $(\mathrm{Id} - \pi_D'(v))^*$ is positive definite, for any $v \notin D$, and the second term in the right hand side of the equality is positive. Therefore,

$$\langle \nabla h(v), \mathbb{B}\nu_D(\pi_D v)\rangle < 0\,, \quad \text{for every } v \notin D\,. \tag{2.5}$$

This implies (1.4), and we see that (2.4) and (2.5) imply (1.5), hence the result follows from Theorem 1.1. □

To end this section, we consider the case when D is a $(N^b + N^c)$-cell, namely

$$D = [a_1, b_1] \times \cdots \times [a_{N^b+N^c}, b_{N^b+N^c}]\,.$$

Corollary 2.4. *Suppose that there exist a* (N^b+N^c)*-uple* $\sigma = (\sigma_1, \dots, \sigma_{N^b+N^c}) \in \{-1,1\}^{N^b+N^c}$ *and a positive constant* $\rho > 0$ *such that, for any solution* $z(t)$ *of* (HS)*, we have*

$$\begin{aligned} &(x_j^b(T) - x_j^b(0))\sigma_j < 0 && \text{if } y_j^b(0) \in [a_j - \rho, a_j]\,, \\ &(x_j^b(T) - x_j^b(0))\sigma_j > 0 && \text{if } y_j^b(0) \in [b_j, b_j + \rho]\,, \\ &(y_k^c(T) - y_k^c(0))\sigma_{N^b+k} < 0 && \text{if } x_k^c(0) \in [a_{N^b+k} - \rho, a_{N^b+k}]\,, \\ &(y_k^c(T) - y_k^c(0))\sigma_{N^b+k} > 0 && \text{if } x_k^c(0) \in [b_{N^b+k}, b_{N^b+k} + \rho]\,, \end{aligned}$$

for every index $j = 1, \dots, N^b$ *and* $k = 1, \dots, N^c$*. Then, the conclusion of Theorem* 1.1 *holds.*

Proof. A smoothness procedure can be used (see [18, Lemma 2.1]) to transform the set D into a strongly convex set with a smooth boundary. Then, taking a diagonal matrix $\mathbb{B}$ with diagonal elements equal to ± 1, Corollary 2.3 applies. □

The expert reader will have noticed that such kind of conditions are strongly related to the ones appearing in the Poincaré–Miranda Theorem.

3. Proof of Theorem 1.1

For simplicity, we assume the Hamiltonian function $H(t,z)$ to be $\mathcal{C}^\infty$. The general case involves some technical arguments which have been presented in [18], and will be avoided here, for briefness.

Let us consider, for any $R \geq 1$, a $\mathcal{C}^\infty$-smooth function $a_R \colon \mathbb{R} \to [0,1]$, with

$$a_R(s) = \begin{cases} 1 & \text{if } s \leq R\,, \\ 0 & \text{if } s \geq 3R\,, \end{cases}$$

and such that

$$-s^{-1} \leq a_R'(s) \leq 0\,, \quad \text{for every } s \geq R\,. \tag{3.1}$$

Gathering together the variables in which $H(t,z)$ is 2π-periodic, let us introduce the notation

$$z^p = (x^a, x^b, y^a, y^c)\,, \qquad z^{\neg p} = (x^c, x^d, y^b, y^d)\,, \tag{3.2}$$

so that we are allowed to write $z = (z^p, z^{\neg p})$. By (1.2) and the periodicity in the z^p variables we can find two constants C_3, C_4 for which

$$|K(t,z)| \leq C_3 |z^{\neg p}| + C_4\,, \qquad \text{for every } z \in \mathbb{R}^{2N}. \tag{3.3}$$

We now define the function

$$K_R(t,z) = a_R(|z^{\neg p}|) K(t,z)\,,$$

for some $R \geq 1$ to be fixed below, and the corresponding Hamiltonian function

$$H_R(t,z) = \tfrac{1}{2}\langle \mathbb{A}(t) z^d, z^d \rangle + K_R(t,z)\,.$$

Notice that, using (1.2), (3.1), (3.3) and the fact that $R \geq 1$, we get

$$\begin{aligned}|\nabla K_R(t,z)| &= \left| a_R'(|z^{\neg p}|) K(t,z) \frac{z^{\neg p}}{|z^{\neg p}|} + a_R(|z^{\neg p}|) \nabla K(t,z) \right| \\ &\leq |a_R'(|z^{\neg p}|)|\, |K(t,z)| + a_R(|z^{\neg p}|)\, |\nabla K(t,z)| \\ &\leq C_1 + C_3 + C_4 := C_5 \,,\end{aligned}$$

so that the bound on ∇K_R is independent of R.

The following lemmas provide us with some a priori estimates on the solutions of

$$\dot z = J \nabla H_R(t,z)\,. \qquad (\mathrm{HS}_R)$$

Lemma 3.1. *There is a constant $C_6 \geq 1$ such that, for any $R \geq 1$, if $z(t)$ is a T-periodic solution of* (HS_R) *satisfying* (1.6)*, then $|z^{\neg p}(t)| \leq C_6$, for every $t \in [0,T]$.*

Proof. Assume by contradiction that there are a sequence $(R_n)_n$ in $[1,+\infty[$ and a sequence $(z_n)_n$ of solutions of (HS_R) with $R = R_n$ such that $\|z_n^{\neg p}\|_\infty \to +\infty$. We first prove that $(z_n^d)_n$ remains bounded. If not, define $v_n(t) = z_n(t)/\|z_n^d\|_\infty$. Then

$$\dot v_n = J\mathbb{A}(t) v_n^d + \frac{1}{\|z_n^d\|_\infty} J \nabla K_{R_n}(t, z_n)\,,$$

and since ∇K_{R_n} is bounded, independently of n, we deduce the existence of a subsequence $(v_{n_k}^d)_k$ which converges uniformly to some continuous function $v^d(t)$. We then see that $v^d(0) = v^d(T)$, and v^d solves $\dot v = J\mathbb{A}(t)v$, hence, by (1.1), $v^d(t)$ has to be identically equal to zero. But this is a contradiction with the fact that $\|v^d\|_\infty = 1$, since $\|v_n^d\|_\infty = 1$, for every n. So, there is a constant C_d for which $|z_n^d(t)| \leq C_d$, for every n and $t \in [0,T]$.

Now, the fact that $(z_n^d)_n$ remains in a compact set and $(x^c(0), y^b(0))$ also belongs to a compact set, combined with the periodicity and the fact that solutions to initial value problems are globally defined, implies that $(x_n^c)_n$ and $(y_n^b)_n$ must be uniformly bounded on $[0,T]$, thus concluding the proof. □

Lemma 3.2. *There is a constant $C_7 \geq 1$ such that, for any $R \geq 1$, if $z(t)$ is a solution of* (HS_R) *and $|z^d(0)| \geq C_7$, then $z^d(0) \neq z^d(T)$.*

Proof. Assume by contradiction that there are a sequence $(R_n)_n$ in $[1,+\infty[$ and a sequence $(z_n)_n$ of solutions of (HS_R) with $R = R_n$ such that $z_n^d(0) = z_n^d(T)$ and $|z_n^d(0)| \to +\infty$. Define $v_n(t) = z_n(t)/\|z_n^d\|_\infty$. Then

$$\dot v_n = J\mathbb{A}(t) v_n^d + \frac{1}{\|z_n^d\|_\infty} J \nabla K_{R_n}(t, z_n)\,,$$

and since ∇K_{R_n} is bounded, independently of n, we deduce the existence of a subsequence $(v_{n_k}^d)_k$ which converges uniformly to some continuous function $v^d(t)$. We then see that $v^d(0) = v^d(T)$ and v^d solves $\dot v = J\mathbb{A}(t)v$, hence, by (1.1), $v^d(t)$ has to be identically equal to zero. But this is a contradiction with the fact that $\|v^d\|_\infty = 1$, since $\|v_n^d\|_\infty = 1$, for every n. □

Set $C^* = \max\{C_6, C_7\} + \rho$.

Lemma 3.3. *There is a $\widetilde{C} \geq C^*$ such that, for any $R \geq 1$, If $z(t)$ is a solution of* (HS_R) *satisfying $|z^{\neg p}(t_0)| \leq C^*$ for some $t_0 \in [0,T]$, then $|z^{\neg p}(t)| \leq \widetilde{C}$, for every $t \in [0,T]$.*

Proof. Just use the fact that $|\nabla K_R|$ is bounded independently of $R \geq 1$, and $\nabla H(t,z)$ has an at most linear growth in z^d. □

We now fix $R \geq \widetilde{C}$. Notice that, with such an R, Lemma 3.1 tells us that the T-periodic solutions of (HS_R) satisfying (1.6) are indeed solutions of (HS).

Let us denote by $\mathcal{Z}\colon \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ the $\mathcal{C}^\infty$-map associating to each couple (t,ζ) the value at time t of the unique solution $\mathcal{Z}(\cdot,\zeta)$ of (HS_R) satisfying $\mathcal{Z}(0,\zeta) = \zeta$. It will be convenient to write $\zeta = (\xi, \eta)$, similarly as above, with

$$\xi = (\xi^a, \xi^b, \xi^c, \xi^d), \quad \eta = (\eta^a, \eta^b, \eta^c, \eta^d).$$

We also use the notation

$$\zeta^a = (\xi^a, \eta^a)\,, \quad \zeta^b = (\xi^b, \eta^b)\,, \quad \zeta^c = (\xi^c, \eta^c)\,, \quad \zeta^d = (\xi^d, \eta^d)\,,$$

and, gathering as in (3.2) the components where the Hamiltonian function is periodic, $\zeta = (\zeta^p, \zeta^{\neg p})$.

For any t, we write $\mathcal{Z}_t := \mathcal{Z}(t,\cdot)\colon \mathbb{R}^{2N} \to \mathbb{R}^{2N}$, so that

(i) $\mathcal{Z}_0$ is the identity map in $\mathbb{R}^{2N}$;
(ii) $\mathcal{Z}_t(\zeta + \vartheta) = \mathcal{Z}_t(\zeta) + \vartheta$, for any $\vartheta = (\vartheta^p, 0) \in 2\pi\mathbb{Z}^M \times \{0\}$;
(iii) each $\mathcal{Z}_t$ is a symplectic $\mathcal{C}^\infty$-diffeomorphism of $\mathbb{R}^{2N}$ on itself.

Now we define the function $\mathfrak{R}\colon \mathbb{R}^{2N} \to \mathbb{R}$ as

$$\mathfrak{R}(\zeta) = -\kappa\, h(\xi^c, \eta^b)\,,$$

for some positive constant κ to be determined later, and consider the Hamiltonian system

$$\dot{z} = J\nabla \widetilde{H}_R(t,z)\,, \tag{$\widetilde{\mathrm{HS}}_R$}$$

with

$$\widetilde{H}_R(t,z) := H_R(t,z) + \mathfrak{R}(\mathcal{Z}_t^{-1}(z))\,.$$

Notice that, if $|z|$ is large enough, then

$$\mathcal{R}(\mathcal{Z}_t^{-1}(z)) = -\kappa h(x^c, y^b)\,, \quad \text{for every } t \in [0,T]\,.$$

The following properties hold:

(j) $\widetilde{H}_R(t, z+w) = \widetilde{H}_R(t,z)$, for any $w = (w^p, 0) \in 2\pi\mathbb{Z}^M \times \{0\}$;
(jj) there is a positive constant $\widetilde{C}$ for which

$$\left|\widetilde{H}_R(t,z) - \tfrac{1}{2}\langle \mathbb{A}(t) z^d, z^d\rangle + \tfrac{1}{2}\kappa\langle \mathbb{S}(x^c, y^b), (x^c, y^b)\rangle\right| \leq \widetilde{C}\,;$$

(jjj) $\widetilde{H}_R$ and H_R coincide on the set

$$\left\{(t, \mathcal{Z}(t,\zeta)) : t \in [0,T],\, |\zeta^{\neg p}| \leq C^*\right\}.$$

It is well known that the T-periodic solutions of $(\widetilde{\mathrm{HS}}_R)$ are the critical points of the functional $\varphi\colon H_T^{1/2} \to \mathbb{R}$, defined as

$$\varphi(z) = \int_0^T \Big[\tfrac{1}{2}\langle J\dot z(t), z(t)\rangle + \widetilde H_R(t, z(t))\Big]\, \mathrm{d}t\,.$$

Denoting by $\bar f$ the mean value of a function $f : [0,T] \to \mathbb{R}$, and defining

$$\vartheta = (\bar x^a, \bar x^b, \bar y^a, \bar y^c)\,,$$

by the periodicity in these variables we can think of ϑ varying on the torus $\mathbb{T}^M$, and the remaining variables

$$v = (\tilde x^a, \tilde x^b, x^c, x^d, \tilde y^a, y^b, \tilde y^c, y^d)$$

belonging to a Hilbert space E (here $\tilde f$ denotes $f - \bar f$). We can thus reduce our search of critical points to an equivalent functional $\widehat\varphi : \mathbb{T}^M \times E \to \mathbb{R}$ of the form

$$\widehat\varphi(\vartheta, v) = \tfrac{1}{2}\langle Lv, v\rangle + \psi(\vartheta, v)\,.$$

To see that L is invertible, consider the system

$$\dot z = J\nabla H_\infty(z)\,, \tag{3.4}$$

with

$$H_\infty(z) = \tfrac{1}{2}\langle \mathbb{A}(t) z^d, z^d\rangle - \tfrac{1}{2}\kappa\langle \mathbb{S}(x^c, y^b), (x^c, y^b)\rangle\,.$$

If $z(t)$ is a T-periodic solution of (3.4), then $z^d \equiv 0$, by (1.1), while

$$(\dot x^b, \dot y^c) = \kappa J\mathbb{S}(x^c, y^b)\,, \qquad (\dot x^c, \dot y^b) = 0\,.$$

Then, x^c and y^b are constant, and since $\mathbb{S}$ is invertible and (x^b, y^c) is periodic, it has to be $x^c \equiv 0$ and $y^b \equiv 0$. Then, x^b and y^c are constant. Since they belong to E, their mean value is zero, hence $x^b \equiv 0$ and $y^c \equiv 0$.

Moreover, it can be seen that ψ' is compact and has a bounded image (cf.[29]). By [30, Theorem 4.2] there are at least $M+1$ critical points, and 2^M in the nondegenerate case [31, Theorem 8.1].

As a consequence of *(jjj)*, the Hamiltonian systems (HS_R) and $(\widetilde{\mathrm{HS}}_R)$ have the same solutions $z(t)$, with $t \in [0,T]$, departing with $(x^c(0), y^b(0)) \in D$ and $|z^d(0)| \le C^*$. Thus, in order to complete the proof, it will suffice to check that $(\widetilde{\mathrm{HS}}_R)$ does not have solutions $\tilde z(t)$, satisfying $\tilde z(0) = \tilde z(T)$, departing with either $(\tilde x^c(0), \tilde y^b(0)) \notin D$, or $|\tilde z^d(0)| > C^*$.

We argue by contradiction, and assume that such a solution $\tilde z(t)$ exists. Let us define the function $\zeta\colon [0,T] \to \mathbb{R}^{2N}$ by

$$\zeta(t) := \mathcal{Z}_t^{-1}(\tilde z(t))\,.$$

We can see that $\zeta(t)$ satisfies

$$\dot\zeta(t) = J\nabla\mathfrak{R}(\zeta(t))\,, \quad \zeta(0) = \tilde z(0)\,.$$

Hence, in particular,

$$(\dot\xi^b(t), \dot\eta^c(t)) = \kappa J\nabla h(\xi^c(t), \eta^b(t))\,, \quad (\dot\xi^c(t), \dot\eta^b(t)) = (0,0)\,, \quad \dot\zeta^d(t) = 0\,.$$

Consequently, writing $\tilde{z}(t) = (\tilde{x}(t), \tilde{y}(t))$, with

$$\tilde{x}(t) = (\tilde{x}^a(t), \tilde{x}^b(t), \tilde{x}^c(t), \tilde{x}^d(t)), \qquad \tilde{y}(t) = (\tilde{y}^a(t), \tilde{y}^b(t), \tilde{y}^c(t), \tilde{y}^d(t)),$$

we have that

$$\xi^c(t) = \xi^c(0) = \tilde{x}^c(0), \quad \eta^b(t) = \eta^b(0) = \tilde{y}^b(0), \quad \zeta^d(t) = \zeta^d(0) = \tilde{z}^d(0),$$
$$(\xi^b(t), \eta^c(t)) = (\tilde{x}^b(0), \tilde{y}^c(0)) - \kappa t\, J\nabla h(\tilde{x}^c(0), \tilde{y}^b(0)),$$

for every $t \in \mathbb{R}$. Being $\tilde{z}(T) = \mathcal{Z}_T(\zeta(T))$, there is a solution $z(t)$ of (HS_R) such that $z(0) = \zeta(T)$ and $z(T) = \tilde{z}(T)$. In particular,

$$(x^c(0), y^b(0)) = (\xi^c(T), \eta^b(T)) = (\xi^c(0), \eta^b(0)) = (\tilde{x}^c(0), \tilde{y}^b(0)),$$
$$z^d(0) = \zeta^d(T) = \zeta^d(0) = \tilde{z}^d(0),$$
$$(x^b(0), y^c(0)) = (\xi^b(T), \eta^c(T)) = (\tilde{x}^b(0), \tilde{y}^c(0)) - \kappa T\, J\nabla h(\tilde{x}^c(0), \tilde{y}^b(0)).$$

Then, since $\tilde{z}(0) = \tilde{z}(T)$,

$$\begin{aligned}(x^b(T) - x^b(0), y^c(T) - y^c(0)) &= (\tilde{x}^b(T), \tilde{y}^c(T)) - \big[(\tilde{x}^b(0), \tilde{y}^c(0)) - \kappa T\, J\nabla h(\tilde{x}^c(0), \tilde{y}^b(0))\big] \\ &= \kappa T\, J\nabla h(x^c(0), y^b(0)).\end{aligned}$$

Moreover, being $z^d(0) = \tilde{z}^d(0) = \tilde{z}^d(T) = z^d(T)$, by Lemma 3.2 it has to be $|z^d(0)| < C_7 \le C^*$ and then, necessarily, $(x^c(0), y^b(0)) \notin D$. Now we consider two cases.

Case 1: $\mathrm{dist}((x^c(0), y^b(0)), \partial D) \le \rho$. Then, by Lemmas 3.1 and 3.3 the solution $z(t)$ is such that $|z^{\neg p}(t)| \le \widetilde{C} \le R$, for every $t \in [0, T]$, hence it is a solution of (HS), and we get a contradiction with (1.5).

Case 2: $\mathrm{dist}((x^c(0), y^b(0)), \partial D) > \rho$. Since D is compact and $\nabla h(v) \neq 0$ for every $v \notin D$, setting

$$c := \inf\{|\nabla h(v)| : \mathrm{dist}(v, D) \ge \rho\} > 0,$$

we have on one hand that

$$\big|(x^b(T) - x^b(0), y^c(T) - y^c(0))\big| \le C_5 T,$$

while, on the other hand,

$$|\kappa T\, J\nabla h(x^c(0), y^b(0))| \ge \kappa T c.$$

Taking $\kappa > C_5/c$, we have a contradiction.

The proof is thus completed.

4. Superintergrable systems and perturbations of low dimensional tori

In [11] it has been shown how, in the special case $N_d = 0$, a generalized version of the Poincaré–Birkhoff Theorem provided in [18] can be applied to perturbations of a completely integrable system, thus generalizing the results in [1, 2, 5, 9], where a sort of periodic counterpart of the celebrated KAM theory was developed. Indeed, it is proved that, whereas tori made of periodic solutions are in general destroyed by a small perturbation, still the survival of a certain number of periodic solutions is guaranteed, assuming some suitable nondegeneracy conditions.

In this section we discuss how such framework can be extended to the case $N_d \geq 1$. There are two main options to do so, depending on whether the linear part of the dynamics is included in the unperturbed system, or corresponds to the lower order terms of the perturbation. We discuss with more detail the latter case, which is slightly more complex. The proof in the two situations is however almost the same; the former case will be briefly commented at the end of the section.

We consider a superintegrable $2N$-dimensional system, namely a Hamiltonian system in $\mathbb{R}^{2N}$ having $2N-M$ constants of motion, for some $0 < M < N$, which are independent and satisfy a suitable rank condition on their Poisson brackets (cf. e.g. [10, 28]), hence producing a foliation in M-dimensional surfaces. By the Mishchenko–Fomenko Theorem [27] we know that, if one of these fibers is compact, then it is an M-torus $\mathbb{T}^M$; moreover, in a neighbourhood of any of such tori the system can be written in the form

$$\begin{cases} \dot{\tilde{x}} = 0 \\ \dot{\varphi} = \nabla \mathcal{K}(I) \\ \dot{\tilde{y}} = 0 \\ \dot{I} = 0\,, \end{cases} \tag{4.1}$$

for some suitable coordinates

$$(\tilde{x}, \varphi, \tilde{y}, I) \in \mathcal{U}_{\tilde{x}} \times \mathbb{T}^M \times \mathcal{U}_{\tilde{y}} \times \mathcal{U}_I \subseteq \mathbb{R}^{N-M} \times \mathbb{T}^M \times \mathbb{R}^{N-M} \times \mathbb{R}^M,$$

and Hamiltonian function $\mathcal{K}(\tilde{x}, \varphi, \tilde{y}, I) = \mathcal{K}(I)$, which we assume to be once continuously differentiable.

We assume that the tori corresponding to a certain value $I = I^0$ in $\mathcal{U}_I$ are composed of T^0-periodic orbits. This means that, denoting by $\omega^0 = (\omega_1^0, \ldots, \omega_M^0) = \nabla \mathcal{K}(I^0)$ the frequency of such tori, there exist M integers $a_1, \ldots, a_M$ such that

$$T^0 \omega_i^0 = 2\pi a_i\,, \quad \text{for every } i = 1, \ldots, M\,. \tag{4.2}$$

We assume moreover that $\mathcal{K}$ is twice differentiable at I^0, and that it satisfies the nondegeneracy condition

$$\det(\mathcal{K}''(I^0)) \neq 0\,. \tag{4.3}$$

If, on one hand, the rich structure of superintegrable systems offers detailed information on the unperturbed system, on the other hands it implies that, in the study of its perturbation, a full nondegeneracy is not available; indeed, since we foliate an open set of $\mathbb{R}^{2N}$ in M-dimensional tori (whose frequency is a vector in $\mathbb{R}^M$), we can always find a lower dimensional fibration of tori with the same frequency. To overcome this issue, and hence to be able to obtain perturbative results such as in KAM theory, one classical approach is to assume that the first order term of the perturbation has a special structure which allows to recover nondegeneracy on the desired directions [10, 20]. We will follow this spirit, although with some differences: indeed we will not use the first order perturbation to produce a foliation in N-tori (to which classical results apply), but instead use it to obtain nondegeneracy on a specific lower dimensional M-torus.

We therefore proceed by considering a perturbation $K_\varepsilon = K_\varepsilon(t, \tilde{x}, \varphi, \tilde{y}, I)$ of $\mathcal{K}$, continuous, T-periodic in t, and continuously differentiable in the other variables. (In what follows, ε will be a small positive parameter.) Using the notation $\tilde{z} = (\tilde{x}, \tilde{y})$, we assume that K_ε is of the form

$$K_\varepsilon(t, \tilde{x}, \varphi, \tilde{y}, I) = \mathcal{K}(I) + \frac{\varepsilon}{2} \langle \mathbb{A}\tilde{z}, \tilde{z} \rangle + \varepsilon^2 P(t, \tilde{x}, \varphi, \tilde{y}, I)\,, \tag{4.4}$$

where $\mathbb{A}$ is a $2(N-M)\times 2(N-M)$ invertible symmetric matrix, while P is T-periodic in time t and its gradient ∇P in the $(\tilde{x}, \varphi, \tilde{y}, I)$ variables is uniformly bounded. Moreover, we assume that there exist two integers m^0, n^0 satisfying

$$n^0 T^0 = m^0 T\,.$$

We are now interested in the study of the perturbed system

$$\begin{cases} \dot{\tilde{x}} = \nabla_{\tilde{y}} K_\varepsilon(t, \tilde{x}, \varphi, \tilde{y}, I) \\ \dot{\varphi} = \nabla_I K_\varepsilon(t, \tilde{x}, \varphi, \tilde{y}, I) \\ \dot{\tilde{y}} = -\nabla_{\tilde{x}} K_\varepsilon(t, \tilde{x}, \varphi, \tilde{y}, I) \\ \dot{I} = -\nabla_\varphi K_\varepsilon(t, \tilde{x}, \varphi, \tilde{y}, I)\,. \end{cases} \tag{4.5}$$

Here is our result.

Theorem 4.1. *For ε sufficiently small, the perturbed system* (4.5) *admits at least $M+1$ geometrically distinct m^0T-periodic solutions, each making exactly $n^0 a_i$ rotations in the coordinate φ_i in one period time, for every $i = 1, \dots, M$.*

Proof. To begin, we would like to extend system (4.5) to $\mathbb{R}^{2N}$. We take $r_0 > 0$ such that $\mathcal{B}_{\mathbb{R}^M}(I_0, r_0) \subseteq \mathcal{U}_I$ and $\mathcal{B}_{\mathbb{R}^{2(N-M)}}(0, r_0) \subseteq \mathcal{U}_{\tilde{x}} \times \mathcal{U}_{\tilde{y}}$, and with the property that

$$\langle \nabla\mathcal{K}(I) - \nabla\mathcal{K}(I^0), \mathcal{K}''(I^0)(I - I^0)\rangle > 0\,, \qquad \text{for every } I \in \mathcal{B}_{\mathbb{R}^M}(I_0, r_0)\,. \tag{4.6}$$

Then, we modify and extend outside these sets the function K_ε defined in (4.4), as follows. Concerning the function $\mathcal{K}(I)$, we can take any smooth extension to $\mathbb{R}^M$ with bounded gradient. Regarding P, we take a continuous extension with uniformly bounded gradient in the variables $\tilde{x}, \tilde{y}, I$ and keep the periodicity in the φ-variables. The linear part of K_ε is extended in the natural way.

In order to apply Corollary 2.3, we need to subtract from the Hamiltonian function a term accounting for the rotation of the reference torus; we thus define

$$K^*_\varepsilon(t, \tilde{x}, \varphi, \tilde{y}, I) = K_\varepsilon(t, \tilde{x}, \varphi, \tilde{y}, I) - \left\langle \nabla\mathcal{K}(I^0), I - I^0 \right\rangle\,.$$

We now study the Hamiltonian system

$$\dot{Z} = J\nabla K^*_\varepsilon(t, Z)\,, \tag{4.7}$$

where $Z(t) = (\tilde{x}(t), \varphi(t), \tilde{y}(t), I(t))$. Any m^0T-periodic solution of this system corresponds to a m^0T-periodic solution of (4.4) making exactly $n^0 a_i$ rotations in each coordinate φ_i in its period time.

We will now prove that there exists $\bar{\varepsilon} > 0$ such that, for every $\varepsilon \in]0, \bar{\varepsilon}[$,

i) the matrix $\varepsilon\mathbb{A}$ satisfies the nonresonance condition

$$\sigma(J\varepsilon\mathbb{A}) \cap \frac{2\pi i}{m^0T}\mathbb{Z} = \emptyset\,;$$

ii) all the m^0T-periodic solutions of (4.7) satisfy

$$\|(\tilde{x}(t), \tilde{y}(t))\| < r_0\,, \quad \text{for every } t \in [0, m^0T]\,;$$

iii) there exists $r_\varepsilon > 0$ such that every solution of (4.7) with $\|I(0) - I_0\| \leq r_\varepsilon$ satisfies

$$\|I(t) - I_0\| \leq r_0\,, \quad \text{for every } t \in [0, m^0T]\,;$$

iv) for every $I \in \mathcal{B}(I_0, r_0)$ and every $t \in \mathbb{R}$, $\varphi \in \mathbb{T}^M$, $(\tilde{x}, \tilde{y}) \in \mathbb{R}^{2(N-M)}$, we have

$$\langle \nabla_I K^*_\varepsilon(t, \tilde{x}, \varphi, \tilde{y}, I), \mathcal{K}''(I_0)(I - I_0)\rangle > 0\,. \tag{4.8}$$

Item i) is easily verified, since $J\mathbb{A}$ is invertible and bounded. It guarantees that (1.1) holds when $\mathbb{A}(t)$ is replaced by $\varepsilon\mathbb{A}$ and T by m^0T.

To check ii), let us notice that the $\tilde{z} = (\tilde{x}, \tilde{y})$-component of each solution satisfies an equation of the type

$$\dot{\tilde{z}} = \varepsilon J\mathbb{A}\tilde{z} + \varepsilon^2 f(t)\,,$$

where f itself depends on the considered solution. Suppose by contradiction that ii) is false. Then there exist some sequences $(\varepsilon_n)_n$, $(f_n)_n$, $(w_n)_n$, with

$$\varepsilon_n \in]0,1[\,, \quad f_n \colon [0, m^0T] \to \mathcal{B}_{\mathbb{R}^{2N-2M}}(0, \|\nabla P\|_\infty)\,, \quad w_n \colon [0, m^0T] \to \mathbb{R}^{2(N-M)}\,,$$

such that $\varepsilon_n \to 0$, $\|w_n\|_\infty \geq r_0$, and

$$\dot{w}_n = \varepsilon_n J\mathbb{A}w_n + \varepsilon_n^2 f_n(t)\,, \qquad w_n(0) = w_n(T)\,. \tag{4.9}$$

If $(\|w_n\|_\infty)_n$ is bounded, then $\|\dot{w}_n\|_\infty \to 0$ and hence, up to a subsequence, $(w_n)_n$ converges uniformly to a constant function $\bar{w}$, with $\|\bar{w}\| \geq r_0$. However, integrating (4.9) over one period, we see that

$$\left\| \int_0^{m^0T} \mathbb{A}w_n(t)\,\mathrm{d}t \right\| \leq \varepsilon_n m^0T\,\|\nabla P\|_\infty\,,$$

and passimg to the limit we conclude that $\mathbb{A}\bar{w} = 0$, which is a contradiction since $\mathbb{A}$ is invertible and $\bar{w} \neq 0$.

If instead $(\|w_n\|_\infty)_n$ is unbounded then, up to a subsequence, $\|w_n\|_\infty \to +\infty$. Then, setting $v_n(t) = w_n(t)/\,\|w_n\|_\infty$, we have that

$$\dot{v}_n = \varepsilon_n J\mathbb{A}v_n + \varepsilon_n^2 \frac{f_n(t)}{\|w_n\|_\infty}\,, \quad v_n(0) = v_n(T)\,. \tag{4.10}$$

Since $(\|v_n\|_\infty)_n$ is bounded, we see that, up to a further subsequence, $(v_n)_n$ converges uniformly to a contant function $\bar v$, with $\|\bar v\| = 1$. As in the previous case we get a contradiction integrating and passing to the limit in

$$\Big\| \int_0^{m^0 T} \mathbb{A} v_n(t)\,\mathrm{d}t \Big\| \le \varepsilon_n m^0 T \frac{\|\nabla P\|_\infty}{\|w_n\|_\infty}\,.$$

Finally, both items iii) and iv) easily follow from the boundedness assumption on the function ∇P, taking into account (4.6).

We now fix $\varepsilon \in]0, \bar\varepsilon[$ and set $D = \mathcal{B}_{\mathbb{R}^M}(I_0, r_\varepsilon/2)$. Integrating (4.8) along the orbits we obtain that the twist condition (2.4) is satisfied with $\mathbb{B} = \mathcal{K}''(I^0)$ and $\rho = r_\varepsilon/2$. By this and i) we can apply Corollary 2.3 and recover $M+1$ geometrically distinct m^0T-periodic solutions of (4.7) satisfying $I(0) \in D$. By ii) and iii), these solutions are such that

$$I(t) \in \mathcal{B}_{\mathbb{R}^M}(I_0, r_0)\,, \quad \tilde z(t) \in \mathcal{B}_{\mathbb{R}^{2N-2M}}(0, r_0)\,, \qquad \text{for every } t \in [0, m^0T]\,,$$

hence they lie in the region where the Hamiltonian function has not been modified. We have thus found $M+1$ distinct m^0T-periodic solutions of (4.5). □

Remark 4.2. The nondegeneracy condition (4.3) at $I = I_0$ can be replaced by the following weaker one (cf. [11]): there exists an invertible symmetric matrix $\mathbb{B}$ such that

$$0 \in \mathrm{cl}\left\{ r \in]0, +\infty[\, : \min_{\|I - I^0\| = r} \big\langle \nabla\mathcal{K}(I) - \nabla\mathcal{K}(I^0), \mathbb{B}(I - I^0)\big\rangle > 0 \right\}. \quad (4.11)$$

Indeed, if (4.3) holds, then (4.11) is satisfied taking $\mathbb{B} = \mathcal{K}''(I^0)$. Notice that such a condition does not require the function $\mathcal{K}$ to be twice differentiable at I^0.

Remark 4.3. The proof of Theorem 4.1 can be easily adapted if in the perturbed system (4.5) we replace the structure (4.4) of the Hamiltonian function by

$$K_\varepsilon(t, \tilde x, \varphi, \tilde y, I) = \mathcal{K}(I) + \tfrac{1}{2}\langle \mathbb{A}\tilde z, \tilde z\rangle + \varepsilon P(t, \tilde x, \varphi, \tilde y, I)\,. \quad (4.12)$$

In this case the linear component is not part of the perturbation, so we have to assume that

$$\sigma(J\mathbb{A}) \cap \frac{2\pi i}{m^0 T}\mathbb{Z} = \emptyset\,,$$

so to guarantee the nonresonance condition (1.1) for $\mathbb{A}$, replacing T by m^0T. The existence of the same number of periodic solutions with the same rotation properties can be proved, for sufficiently small ε, in the same way as in Theorem 4.1. Concerning the survival of KAM tori for perturbations of the type (4.12), we refer to [3, Section 8.3] and references therein.

5. Weakly coupling linear and superlinear systems

Let us consider the system

$$\begin{cases} \ddot{u} + \mathbb{M}(t)u = \nabla_u \mathcal{U}(t,u,w)\,, \\ \ddot{w} + \nabla V(t,w) = \nabla_w \mathcal{U}(t,u,w)\,, \end{cases} \tag{5.1}$$

where $u = (u_1, \dots, u_l) \in \mathbb{R}^l$, $w = (w_1, \dots, w_m) \in \mathbb{R}^m$. Here $\mathbb{M}(t)$ is a symmetric matrix, continuous and T-periodic in time, satisfying the nonresonance condition

$$u(t) \equiv 0 \quad \text{is the only } T\text{-periodic solution of } \ddot{u} = \mathbb{M}(t)u\,. \tag{5.2}$$

We assume that

$$V(t,w) = \sum_{k=1}^{M} V_k(t,w_k)\,, \qquad \text{with} \qquad V_k(t,s) = \int_0^s \sigma h_k(t,\sigma)\,\mathrm{d}\sigma\,,$$

where the functions $h_k \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous, T-periodic in t, and satisfy

$$\lim_{|\sigma| \to +\infty} h_k(t,\sigma) = +\infty\,, \quad \text{uniformly in } t \in [0,T]. \tag{5.3}$$

The second equation in (5.1) then reads as

$$\begin{cases} \ddot{w}_1 + w_1 h_1(t,w_1) = \frac{\partial}{\partial w_1}\mathcal{U}(t,u,w)\,, \\ \vdots \\ \ddot{w}_M + w_M h_M(t,w_M) = \frac{\partial}{\partial w_M}\mathcal{U}(t,u,w)\,. \end{cases} \tag{5.4}$$

The function $\mathcal{U} \colon \mathbb{R} \times \mathbb{R}^l \times \mathbb{R}^m$ is continuous, T-periodic in t, continuously differentiable in (u,w) and satisfies, for some $C > 0$,

$$|\nabla_u \mathcal{U}(t,u,w)| < C\,, \tag{5.5}$$

$$\frac{\partial}{\partial w_k}\mathcal{U}(t,u,w) = w_k p_k(t,u,w)\,, \text{ with } |p_k(t,u,w)| < C\,, \tag{5.6}$$

for every $(t,u,w) \in [0,T] \times \mathbb{R}^l \times \mathbb{R}^m$. We observe that, thanks to the structure induced by $\mathcal{U}$, system (5.1) can be rewritten as a Hamiltonian system on $\mathbb{R}^{2(l+m)}$, with variables $(u, w, \dot{u}, \dot{w})$.

We have the following result.

Theorem 5.1. *There exists a positive integer K such that, for every choice of m integers $\ell_1, \dots, \ell_m \geq K$, system* (5.1) *has at least $m+1$ solutions which are T-periodic and such that each of the components w_k has exactly $2\ell_k$ simple zeros in the interval* $[0,T)$.

The above theorem extends the result in [15], where a classical theorem by Jacobowitz [22] and Hartman [21] for scalar equations was generalized for the superlinear system (5.4). The special form of the nonlinearities is needed here so to guarantee that the "unperturbed system" (5.4), with $\mathcal{U} \equiv 0$, has the trivial solutions $w_k \equiv 0$. This permits on one side to prove the global existence of the solutions, and on the other side to settle each equation in polar coordinates. Notice that the same approach can be followed if we replace each $w_k h_k(t, w_k)$ by some $g_k(w_k)$, not depending on time, as shown in [7] for the scalar equation, and in [18] for a system.

In dealing with the superlinear part, we follow the classical approach adopted in [15]. First of all, since our method is based on the study of the Poincaré time map on a suitable portion of the domain, we want to assure the global existence of solutions for the Cauchy problems. To do so, we in fact consider, for every $R > 1$, the auxiliary system

$$\begin{cases} \ddot{u} + \mathbb{M}(t)u = \nabla_u \mathcal{U}(t, u, w)\,, \\ \ddot{w}_1 + w_1 h^R(t, w_1) = \frac{\partial}{\partial w_1}\mathcal{U}(t, u, w)\,, \\ \vdots \\ \ddot{w}_1 + w_M h^R(t, w_M) = \frac{\partial}{\partial w_M}\mathcal{U}(t, u, w)\,, \end{cases} \tag{5.7}$$

where the functions h_k^R are defined as

$$h_k^R(t, w_k) = \begin{cases} h_k(t, -R)\,, & \text{if } w_k < -R\,, \\ h_k(t, w_k)\,, & \text{if } |w_k| \le R\,, \\ h_k(t, R)\,, & \text{if } w_k > R\,. \end{cases}$$

Let us study a solution (u, w) of system (5.7). For every $k = 1, \dots, m$, let us consider the orbit $(w_k(t), \dot{w}_k(t))$ in the phase plane and assume that $(w_k(t), \dot{w}_k(t)) \neq (0, 0)$ for every $t \in [\tau_0, \tau_1]$. Then this couple can be written in polar coordinates as

$$w_k(t) = \rho_k(t)\cos\vartheta_k(t)\,, \quad \dot{w}_k(t) = \rho_k(t)\sin\vartheta_k(t)\,, \tag{5.8}$$

with $\rho_k(t) > 0$ and $\vartheta_k(t)$ continuous, thus defining

$$\operatorname{rot}_k(u, w, [\tau_0, \tau_1]) = -\frac{\vartheta_k(\tau_1) - \vartheta_k(\tau_0)}{2\pi}\,.$$

We write $\operatorname{rot}_k(u, w) := \operatorname{rot}_k(u, w, [0, T])$. We observe that, if $w_k(t)$ is T-periodic, then $\operatorname{rot}_k(u, w)$ is the integer counting the number of clockwise rotations performed by $(w_k(t), \dot{w}_k(t))$ around the origin, and $2\operatorname{rot}_k(u, w)$ is the number of simple zeros of w_k in the time interval $[0, T)$.

Our approach is based on the following corollary of Theorem 1.1. For $0 < R_1 < R_2$ we introduce the planar annulus $\mathcal{A} = \mathcal{B}[0, R_2] \setminus \mathcal{B}(0, R_1) \subseteq \mathbb{R}^2$; we then set $\Omega = \{(w, \dot{w}) \in \mathbb{R}^{2m} \mid (w_k, \dot{w}_k) \in \mathcal{A}, k = 1, \dots, m\}$.

Corollary 5.2. *Suppose that there exist m positive integers $\ell_1, \dots, \ell_m$ and a constant $\bar{\rho} > 0$ such that, for $k = 1, \dots, m$, we have*

$$\operatorname{rot}_k(u, w) \le \ell_k \qquad \textit{if } \sqrt{w_1(0)^2 + \dot{w}_k(0)^2} \in [R_1 - \bar{\rho}, R_1]\,, \tag{5.9}$$

$$\operatorname{rot}_k(u, w) \ge \ell_k \qquad \textit{if } \sqrt{w_1(0)^2 + \dot{w}_k(0)^2} \in [R_2, R_2 + \bar{\rho}]\,, \tag{5.10}$$

where (u, w) is any solution of (5.7). Then the system (5.7) has $m + 1$ solutions, T-periodic in time, such that each of their components w_k has exactly $2\ell_k$ simple zeros in the interval $[0, T)$.

Proof. Let us begin by observing that system (5.7) is equivalent to the Hamiltonian system on $\mathcal{R}^{2(l+m)}$ with Hamiltonian function

$$\mathfrak{h}(t,u,w,\dot u,\dot w) = \tfrac{1}{2}(|\dot u|^2 + |\dot w|^2) + \tfrac{1}{2}\langle \mathbb{M}(t)u,u\rangle + \sum_{i=1}^{m} V_k^R(t,w_k) + \mathcal{U}(t,u,w)\,,$$

where $V_k^R(t,s) = \int_0^s \sigma h_k^R(t,\sigma)\,\mathrm{d}\sigma$.

By (5.6) and by the continuity of h_R, there exists a constant $a>0$ such that

$$\left|h_k^R(t,w_k) + p_k(t,u,w)\right| < a\,,$$
$$\text{for every } (t,u,w)\in[0,T]\times\mathbb{R}^l\times\mathbb{R}^m,\ w_k\in[-1,1]\,. \tag{5.11}$$

Recalling the polar coordinates introduced in (5.8), we observe that (5.11) implies

$$|\dot\rho_k(t)| \le (a+1)\rho_k(t)\,,$$

along every arc of solution $(u(t),w(t))$ of (5.7). A standard application of Gronwall Lemma provides the existence of a positive constant $\bar\delta<1$ such that every solution of (5.7) with initial point such that $(w_k(0),\dot w_k(0))\in\mathcal{A}$ (hence $\rho_k\ge R_1$) satisfies $\rho_k(t)>2\bar\delta$ for every $t\in[0,T]$ and index $i=1,\dots,m$.

It is therefore allowed to modify the Hamiltonian function $\mathfrak{h}$ in the cylinder $\{(t,u,w,\dot u,\dot w)\mid w_k^2+\dot w_k^2<4\bar\delta^2, i=1,\dots,m\}$ without affecting the solutions with $(w(0),\dot w(0))\in\Omega$. Hence, given any $\psi\in\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ such that

$$\psi(r) = \begin{cases} 0 & \text{if } r\le\bar\delta\,,\\ 1 & \text{if } r\ge 2\bar\delta\,,\end{cases}$$

we may study the new system with Hamiltonian

$$\mathfrak{h}_0(t,u,w,\dot u,\dot w) = \mathfrak{h}(t,u,w,\dot u,\dot w)\,\psi\Big(\min\Big\{\sqrt{w_k^2+\dot w_k^2}\,, i=1,\dots,m\Big\}\Big).$$

We now consider the time-dependent change of variables

$$u_k = x_k^d \qquad\qquad \dot u_k = y_k^d$$
$$w_k = \sqrt{2y_k^b}\,\cos\left(x_k^b - \frac{2\pi\nu_k t}{T}\right) \qquad \dot w_k = \sqrt{2y_k^b}\,\sin\left(x_k^b - \frac{2\pi\nu_k t}{T}\right)$$

in order to obtain, when $y_k^b\ge 0$, a Hamiltonian system (HS) with $N^b=m$, $N^d=l$, $N_a=N_c=0$ and

$$H(t,z) = \mathfrak{h}_0(t,u,w,\dot u,\dot w) + \sum_{k=1}^{N^b}\frac{2\pi\nu_k y_k^b}{T}\,.$$

We can extend this Hamiltonian function for $y_k^b<0$ by simply setting $H(t,z)=\sum_{k=1}^{N^b}2\pi\nu_k y_k^b/T$ in such situations. We notice that this system satisfies all the assumptions of Corollary 2.4, with the twist conditions obtained by (5.9) and (5.10) with $D=[R_1^2/2,R_2^2/2]^m$. We conclude the proof observing that the $m+1$ periodic solutions given by Theorem 1.1, once translated in the original coordinates, provide the solutions we are looking for. □

The main task to accomplish in order to apply Corollary 5.2 is to provide suitable estimates of the rotational properties of $(w_k(t), \dot{w}_k(t))$. Such situations are well studied for superlinear second order ODEs, and can be extended to systems by showing a suitable uniform behaviour with respect to the other variables. In doing this we follow and generalize the approach in [15], where the case $l = 0$ was treated, providing analogous estimates for the case $l > 0$.

We need the following three lemmas.

Lemma 5.3. *There exists a positive integer K and a positive constant δ such that if a solution (u, v) of (5.7) satisfies $0 < w_k(t_0)^2 + \dot{w}_k(t_0)^2 < \delta^2$ for a certain index k at some time t_0, then*

$$0 < w_k(t)^2 + \dot{w}_k(t)^2 < 1 \,, \quad \text{for every } t \in [t_0, t_0 + T] \,,$$
$$\operatorname{rot}_k(u, w, [t_0, t_0 + T]) < K \,.$$

Moreover such constants do not depend on the value of $R > 1$.

Proof. Let us observe that, since $R > 1$, the constant a obtained in the estimate (5.11) does not depend on R. Using this estimate, and remarking that it is uniform in the variables u, the lemma can be proved with the very same argument of [15, Lemma 3]. □

Lemma 5.4. *For every positive integer K_0, there exists a constant $\mathcal{R} = \mathcal{R}(K_0) > 1$ such that, given a solution (u, v) of (5.7), with $R > \mathcal{R}$, satisfying $\operatorname{rot}_k(u, w) \le K_0$ for every $k = 1, \dots, m$, we have*

$$w_k(t)^2 + \dot{w}_k(t)^2 < \mathcal{R}^2, \quad \text{for every } t \in [0, T] \text{ and every } k = 1, \dots, m \,. \tag{5.12}$$

Proof. Let us fix a constant b, with

$$b > 2\Big(\frac{2\pi K_0}{T}\Big)^2 .$$

By (5.3) and (5.6) we obtain the existence of two constants c and $\bar{R}$, with $1 < c < \bar{R}$, such that, for every index $k = 1, \dots, m$ and for every $R > \bar{R}$, we have

$$h_k^R(t, w_k) + p_k(t, u, w) > h_k^R(t, w_k) - C > b \,,$$
$$\text{for every } (t, u, w) \in [0, T] \times \mathbb{R}^l \times \mathbb{R}^m, \ |w_k| \ge c \,. \tag{5.13}$$

Furthermore, by continuity, there exists a constant $\mathcal{M} > 0$ such that, for every index $k = 1, \dots, m$,

$$\big| w_k \big[h_k^R(t, w_k) + p_k(t, u, w) \big] \big| < \mathcal{M} \,,$$
$$\text{for every } (t, u, w) \in [0, T] \times \mathbb{R}^l \times \mathbb{R}^m, \ |w_k| \le c \,. \tag{5.14}$$

As for the previous lemma, once we have such uniform estimates, the proof can now be concluded following [15, Lemma 4]. □

Lemma 5.5. *Given any positive integer K_0, for every solution (u, v) of (5.7), with $R > \mathcal{R}(K_0)$, such that $w_k(0)^2 + \dot{w}_k(0)^2 = \mathcal{R}^2$ for some index k, we have that $\operatorname{rot}_k(u, w) > K_0$.*

The lemma is proved as in [15, Lemma 6]. We are now ready to prove our result.

Proof of Theorem 5.1. Let us take the integer K and the constant δ provided by Lemma 5.3. We now fix m integers $\ell_1, \dots, \ell_m$ and choose K_0 such that $K_0 \geq \ell_k \geq K$ for every $k = 1, \dots, m$. We apply Lemma 5.4 to recover the corresponding constant $\mathcal{R} = \mathcal{R}(K_0)$. We are now able to fix the constant $R > \mathcal{R}$ and to apply Corollary 5.2 to the associated system (5.7). Indeed, we observe that setting $R_1 := \delta$, condition (5.9) is fulfilled by Lemma 5.3; whereas setting $R_2 := \mathcal{R}$ Lemma 5.5 implies condition (5.10).

To show that the $m+1$ periodic solutions thus provided by Corollary 5.2 for system (5.7) are also solutions of system (5.1), we observe that, by Lemma 5.4, their orbits are contained in the region $w_k^2 + \dot{w}_k^2 < \mathbb{R}^2$, for $k = 1, \dots, m$, where the two systems coincide. □

As a simple example of application of the above result, we can consider the system

$$\begin{cases} \ddot{x}_1 + a(t)x_1 = \frac{\partial}{\partial x_1}\mathcal{U}(t, x_1, x_2)\,, \\ \ddot{x}_2 + x_2^3 = \frac{\partial}{\partial x_2}\mathcal{U}(t, x_1, x_2)\,. \end{cases}$$

The assumptions on $a(t)$ and $\mathcal{U}(t, x_1, x_2)$ can be easily recovered, so to apply Theorem 5.1. We avoid the details, for briefness.

6. Further applications

In the previous section we have treated in detail the coupling of linear and superlinear second order equations. Let us now briefly argue on the possible applications of Theorem 1.1 to other situations involving the coupling of a linear system with a twisting one.

When the second system has a sublinear growth at infinity, we can use the approach developed in [8, 17] to get an infinite number of subharmonic solutions, i.e., periodic solutions having as minimal period an integer multiple of T. Indeed, the large amplitude solutions of the second system rotate around the origin very slowly, with a time of rotation going to infinity with the amplitude. Passing to polar coordinates, we recover the necessary twist, and Corollary 5.2 can be applied. As an example, we could deal with the following system:

$$\begin{cases} \ddot{x}_1 + a(t)x_1 = \frac{\partial}{\partial x_1}\mathcal{U}(t, x_1, x_2)\,, \\ \ddot{x}_2 + \arctan x_2 = \frac{\partial}{\partial x_2}\mathcal{U}(t, x_1, x_2)\,. \end{cases}$$

Another possibility is to couple a linear system with a pendulum-like system. For this type of system there is a large literature, cf. [11, 18] and the references therein. As an example, we could have the following:

$$\begin{cases} \ddot{x}_1 + a(t)x_1 = \frac{\partial}{\partial x_1}\mathcal{U}(t, x_1, x_2)\,, \\ \ddot{x}_2 + \sin x_2 = \frac{\partial}{\partial x_2}\mathcal{U}(t, x_1, x_2)\,. \end{cases}$$

For this kind of equations the twist can be recovered in two different ways. First, writing the equivalent system in (x_2, y_2), with $y_2 = \dot{x}_2$, it is easy to see that solutions starting with $y_2(0)$ large and positive will be such that $x_2(T) > x_2(0)$, while those starting with $y_2(0)$ large and negative will satisfy $x_2(T) < x_2(0)$, so that we have the desired twisting property. Second, one observes that the solutions near the origin of the unperturbed problem are periodic, and their periods increase to infinity when the solutions approach the heteroclinic orbit connecting $(-\pi, 0)$ with $(\pi, 0)$. Then, passing to polar coordinates, the twist is preserved under small perturbations (see e.g. [11]).

A similar argument applies when coupling a linear system with one having different rotational properties near the origin and near infinity (see [25] for a precise description of the twisting properties in this case), or for systems involving a parameter (see, e.g., [12] and the references therein), where the same situation is recovered after a change of coordinates.

To conclude, let us mention the possibility of treating, with the same techniques, problems with one or more singularities. We refer to [16] for the details concerning the singular system, to be coupled with a linear one.

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