

# Periodic perturbations with rotational symmetry of planar systems driven by a central force

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**Abstract.** We consider periodic perturbations of a central force field having a rotational symmetry, and prove the existence of nearly circular periodic orbits. We thus generalize, in the planar case, some previous bifurcation results obtained by Ambrosetti and Coti Zelati in [1]. Our results apply, in particular, to the classical Kepler problem.

## 1 Introduction

We study the planar system

$$\ddot{x} + g(|x|)x = \varepsilon \nabla_x V(t, x), \quad (S_\varepsilon)$$

where  $g : ]0, +\infty[ \rightarrow \mathbb{R}$  is a continuously differentiable function and  $V : \mathbb{R} \times (\mathbb{R}^2 \setminus \{0\}) \rightarrow \mathbb{R}$  is continuous,  $T$ -periodic in its first variable, and twice continuously differentiable in its second variable. The real number  $\varepsilon$  will be considered as a small parameter.

A particularly important special case of  $(S_\varepsilon)$  is illustrated by the perturbed Kepler problem

$$\ddot{x} + \frac{c}{|x|^3}x = \varepsilon \nabla_x V(t, x), \quad (K_\varepsilon)$$

where  $c$  is a positive constant. This kind of systems has been studied by plenty of authors, mainly with the aim of finding periodic solutions: see, e.g., [1, 2, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 17, 19]. Let us describe two of these results in detail.

In 1989, Ambrosetti and Coti Zelati [1] considered system  $(K_\varepsilon)$  in any dimension, and proved that periodic solutions of period  $2T$  exist assuming the perturbing force to be *even* in  $x$ , i.e., that

$$V(t, x) = V(t, -x), \quad \text{for every } t, x.$$

Their method of proof is variational, and provides a bifurcation result from a circular solution  $x_*(t)$  of  $(K_0)$  having minimal period  $\tau_* = 2T/n$ , with  $n$  odd. (By *circular solution* we mean a solution whose orbit is a circle centered at the origin.) Indeed, they proved that at least three such solutions  $x(t)$  exist, satisfying the symmetry property

$$x(t + T) = -x(t), \quad \text{for every } t.$$

Recently, the authors of the present paper considered in [10] the case when the perturbing force is *radially symmetric*, i.e., when  $\nabla_x V(t, x) = p(t, |x|)x$ , for some scalar function  $p : \mathbb{R} \times ]0, +\infty[ \rightarrow \mathbb{R}$ . It was proved that, if  $x_*(t)$  is a circular solution having minimal period  $\tau_*$ , and  $T/\tau_*$  is a rational number  $n/m$  which is not an integer, then near this circular solution there are  $mT$ -periodic solutions  $x(t)$  of system  $(K_\varepsilon)$  making exactly  $n$  rotations around the origin in their period time, provided that  $|\varepsilon|$  is small enough. Moreover,  $t \mapsto |x(t)|$  is  $T$ -periodic, and  $t \mapsto \text{Rot}(x, [t, t + T])$  is constant.

Let us recall here that, when  $\gamma : [\tau_1, \tau_2] \rightarrow \mathbb{R}^2$  is a curve such that  $\gamma(t) \neq 0$  for every  $t$ , writing  $\gamma(t) = \rho(t)(\cos \theta(t), \sin \theta(t))$ , all functions being continuous, the *rotation number* around the origin is defined as

$$\text{Rot}(\gamma; [\tau_1, \tau_2]) = \frac{\theta(\tau_2) - \theta(\tau_1)}{2\pi}.$$

In this paper we assume that the perturbing force has a *rotational symmetry*, i.e., there is a rotation around the origin  $\mathcal{R}$ , of angle

$$\vartheta = \frac{2\pi}{m},$$

with  $m$  being a positive integer, such that

$$\nabla_x V(t, \mathcal{R}x) = \mathcal{R}\nabla_x V(t, x), \quad \text{for every } t, x. \quad (1)$$

Notice that, if  $m = 1$ , the above assumption will be trivially satisfied.

Let  $\rho_* > 0$  be such that  $g(\rho_*) > 0$ . Then, the unperturbed equation  $(S_0)$  has a circular solution  $x_*(t)$ , with  $|x_*(t)| = \rho_*$ , having minimal period

$$\tau_* = \frac{2\pi}{\sqrt{g(\rho_*)}}. \quad (2)$$

We will ask the quotient  $T/\tau_*$  to be a rational number; more precisely, we assume that there exist some positive integers  $\ell, n$  for which

$$\frac{mT}{\tau_*} = \frac{n}{\ell}. \quad (3)$$

We then look for solutions  $x : \mathbb{R} \rightarrow \mathbb{R}^2 \setminus \{0\}$  such that

$$x(t + \ell T) = \mathcal{R}^n x(t), \quad \text{for every } t; \quad (4)$$

these solutions are  $m\ell T$ -periodic, and rotate  $n$  times around the origin in their period time. Notice that, in this case,  $t \mapsto |x(t)|$  is  $\ell T$ -periodic and  $t \mapsto \text{Rot}(x, [t, t + \ell T])$  is constant, precisely equal to  $n/m$ .

It will be useful to introduce the auxiliary function

$$h(\rho) = 4g(\rho) + g'(\rho)\rho.$$

Here is our main result.

**Theorem 1.** *In the above setting, assume that  $g'(\rho_*) \neq 0$  and*

$$h(\rho_*) \notin \left\{ \left( \frac{2\pi k}{\ell T} \right)^2 : k \in \mathbb{N} \setminus \{0\} \right\}. \quad (5)$$

*Then, for any  $\sigma > 0$  there exists  $\bar{\varepsilon} > 0$  such that, if  $|\varepsilon| \leq \bar{\varepsilon}$ , system  $(S_\varepsilon)$  has at least four geometrically distinct  $m\ell T$ -periodic solutions  $x(t)$  satisfying (4). Two of them rotate counter-clockwise, and are such that, for some  $\theta_1, \theta_2$  in  $[0, \tau[$ ,*

$$|x(t) - x_*(t + \theta_i)| \leq \sigma, \quad \text{for every } t \in \mathbb{R}. \quad (6)$$

*The other two solutions rotate clockwise, and are such that, for some  $\theta_3, \theta_4$  in  $[0, \tau[$ ,*

$$|x(t) - x_*(-t + \theta_i)| \leq \sigma, \quad \text{for every } t \in \mathbb{R}. \quad (7)$$

When assuming (1), once a solution  $x(t)$  has been found, also  $\mathcal{R}^j x(t)$  is clearly a solution, with  $j = 1, \dots, m - 1$ . In the above statement, we say that the solutions are *geometrically distinct* if they are not related to each other in this way.

When dealing with the perturbed Kepler problem  $(K_\varepsilon)$ , the functions  $h(\rho)$  and  $g(\rho) = c\rho^{-3}$  coincide, and we have following.

**Corollary 2.** *If (3) is verified, and*

$$\frac{\ell T}{\tau_*} = \frac{n}{m} \notin \mathbb{N}, \quad (8)$$

*then the same conclusion of Theorem 1 holds for system  $(K_\varepsilon)$ .*

Indeed, in this case we have that  $h(\rho_*) = (2\pi/\tau_*)^2$ , so that (5) follows directly from (8). Notice that the case  $\ell = 1$ ,  $m = 2$ , and  $n$  odd was treated in [1, Theorem 2.2], showing the existence of *at least three*  $2T$ -periodic solutions near  $x_*(t)$ , a circular solution of the unperturbed problem having minimal period  $\tau_* = 2T/n$ , in an arbitrary dimensional space. In the planar case, there are at least four  $m\ell T$ -periodic solutions because it is possible to distinguish between two of them rotating clockwise and two counter-clockwise. On the other hand, when considering the radially symmetric situation as in [10], the integer number  $m \geq 2$  can be chosen arbitrarily. Theorem 1 thus extends [10, Corollary 4].

Now, since the minimal period of the circular orbits of the Kepler problem is strictly increasing with their radius, we may expect that, taking *any* circular orbit  $x_*(t)$  of the unperturbed system  $(K_0)$ , the assumptions of Corollary 2 will be satisfied by many circular orbits which are arbitrarily near to it. This is indeed the case, and will be established in Theorem 6 below.

The paper is organized as follows. In Section 2 we provide the proof of Theorem 1 by the use of a bifurcation result, in a variational setting. In Section 3 we consider a generalized Kepler problem, taking  $g(\rho) = c\rho^{-\gamma}$ , for some  $\gamma > 0$ , and we provide the existence of a myriad of periodic solutions near *any* circular solution of the unperturbed system. As an example of application we propose a gravitational model of a pulsating star having some rotational symmetry.

Let us also mention that all our results still hold when  $V(t, x)$  is replaced by some  $V(t, x, \varepsilon)$ , with a smooth dependence on the parameter  $\varepsilon$ .

## 2 Proof of Theorem 1

Searching for  $m\ell T$ -periodic solutions, we will deal with the Hilbert space

$$H_{m\ell T}^1 = \{x \in H^1(0, m\ell T) : x(0) = x(m\ell T)\},$$

equipped with the usual scalar product

$$\langle v, w \rangle = \int_0^{m\ell T} [\langle v(t), w(t) \rangle + \langle \dot{v}(t), \dot{w}(t) \rangle] dt.$$

The elements of this space will be identified to the  $m\ell T$ -periodic continuous functions with locally square integrable weak derivative.

In order to take into account the symmetry requirement (4), we will work on the Hilbert subspace

$$\mathcal{E} = \{x \in H_{m\ell T}^1 : x(t + \ell T) = \mathcal{R}^n x(t), \text{ for every } t \in [0, m\ell T]\}.$$

Here, we recall,  $\mathcal{R}$  is a rotation around the origin of angle  $2\pi/m$ . Without loss of generality, we will assume it to be counter-clockwise, i.e., in complex notation, if  $z \in \mathbb{R}^2$ ,

$$\mathcal{R}z = e^{\frac{2\pi i}{m}} z.$$

Moreover, since we are interested in solutions which never cross the origin, it will be worth introducing the open subset

$$\mathcal{E}_* = \{x \in \mathcal{E} : x(t) \neq 0, \text{ for every } t \in [0, m\ell T]\}.$$

We look for the critical points of the functional  $\varphi_\varepsilon : \mathcal{E}_* \rightarrow \mathbb{R}$  defined as

$$\varphi_\varepsilon(x) = \int_0^{m\ell T} \left[ \frac{1}{2} |\dot{x}(t)|^2 - F(|x(t)|) + \varepsilon V(t, x(t)) \right] dt,$$

where

$$F(\rho) = \int_0^\rho sg(s) ds.$$

Indeed, we have the following information.

**Lemma 3.** *Under the assumption (1), the critical points of the functional  $\varphi_\varepsilon$  correspond to  $m\ell T$ -periodic solutions of  $(S_\varepsilon)$  satisfying (4).*

*Proof.* Let  $x \in \mathcal{E}_*$  be a critical point of  $\varphi_\varepsilon$ , namely

$$\varphi'_\varepsilon(x)v = \int_0^{m\ell T} [\langle \dot{x}(t), \dot{v}(t) \rangle - \langle g(|x(t)|)x(t) - \varepsilon \nabla_x V(t, x(t)), v(t) \rangle] dt = 0,$$

for every  $v \in \mathcal{E}$ . Notice that, since both  $x$  and  $v$  belong to  $\mathcal{E}$ , using (1) we have that

$$\begin{aligned} & \int_0^{m\ell T} [\langle \dot{x}(t), \dot{v}(t) \rangle - \langle g(|x(t)|)x(t) - \varepsilon \nabla_x V(t, x(t)), v(t) \rangle] dt = \\ &= \sum_{j=0}^{m-1} \int_{j\ell T}^{(j+1)\ell T} [\langle \dot{x}(t), \dot{v}(t) \rangle - \langle g(|x(t)|)x(t) - \varepsilon \nabla_x V(t, x(t)), v(t) \rangle] dt \\ &= \sum_{j=0}^{m-1} \int_0^{\ell T} [\langle \dot{x}(t + j\ell T), \dot{v}(t + j\ell T) \rangle - g(|x(t + j\ell T)|) \langle x(t + j\ell T), v(t + j\ell T) \rangle \\ & \quad - \varepsilon \langle \nabla_x V(t + j\ell T, x(t + j\ell T)), v(t + j\ell T) \rangle] dt \\ &= m \int_0^{\ell T} [\langle \dot{x}(t), \dot{v}(t) \rangle - \langle g(|x(t)|)x(t) - \varepsilon \nabla_x V(t, x(t)), v(t) \rangle] dt. \end{aligned}$$

Hence,

$$\int_0^{\ell T} [\langle \dot{x}(t), \dot{v}(t) \rangle - \langle g(|x(t)|)x(t) - \varepsilon \nabla_x V(t, x(t)), v(t) \rangle] dt = 0,$$

for every  $v \in H_{\ell T}^1$ , since such a function  $v(t)$  can be extended on  $[0, m\ell T]$  to a function belonging to the space  $\mathcal{E}$ . We then deduce that  $\dot{x}(t)$  has a weak derivative on  $[0, \ell T]$ , which almost everywhere coincides with  $\psi(t) = -g(|x(t)|)x(t) + \varepsilon \nabla_x V(t, x(t))$ , a continuous function. Therefore,  $x(t)$  is twice continuously differentiable on  $[0, \ell T]$ , and it is a classical solution of the differential equation  $(S_\varepsilon)$  on  $[0, \ell T]$ . Since  $x(t + \ell T) = \mathcal{R}^n x(t)$  and (1) holds, the same conclusion is reached on every interval  $[j\ell T, (j+1)\ell T]$ , with  $j = 1, \dots, m-1$ . Being  $\psi(t)$  continuous on the whole real line, we see that  $x(t)$  is indeed a classical  $m\ell T$ -periodic solution of  $(S_\varepsilon)$ .  $\square$

In order to state the abstract perturbation result, let us consider the symmetric bilinear form  $\varphi_0''(x)$  which associates to every pair of vectors  $v, w \in \mathcal{E}$  the real number

$$\begin{aligned} \varphi_0''(x)(v, w) &= \lim_{\lambda \rightarrow 0} \frac{\varphi_0'(x + \lambda w)v - \varphi_0'(x)v}{\lambda} \\ &= \int_0^{m\ell T} \left[ \langle \dot{v}, \dot{w} \rangle - \frac{g'(|x|)}{|x|} \langle x, v \rangle \langle x, w \rangle - g(|x|) \langle v, w \rangle \right] dt \end{aligned}$$

By the Riesz representation theorem, we associate to this bilinear form a bounded selfadjoint operator in  $\mathcal{E}$ , still denoted by  $\varphi_0''(x)$ , such that, for every  $v, w \in \mathcal{E}$ ,

$$\langle \varphi_0''(x)v, w \rangle = \varphi_0''(x)(v, w).$$

Let us now recall a well-established perturbation result which will be used in our proof (see, e.g., [3, 7, 15, 16, 18]).

**Theorem 4.** *Let an open subset  $\mathcal{E}_*$  of a Hilbert space  $\mathcal{E}$  be the domain of a family of twice continuously differentiable functionals  $\varphi_\varepsilon : \mathcal{E}_* \rightarrow \mathbb{R}$ , depending smoothly on  $\varepsilon$ . Let  $\mathcal{Z} \subseteq \mathcal{E}_*$  be a compact manifold without boundary such that*

- (i)  $\varphi_0'(x) = 0$ , for every  $x \in \mathcal{Z}$ ;
- (ii)  $\varphi_0''(x)$  is a Fredholm operator of index 0, for every  $x \in \mathcal{Z}$ ;
- (iii)  $T_x \mathcal{Z} = \ker \varphi_0''(x)$ , for every  $x \in \mathcal{Z}$ .

*Then, there exists  $\bar{\varepsilon} > 0$  such that, if  $|\varepsilon| \leq \bar{\varepsilon}$ , the functional  $\varphi_\varepsilon$  has at least  $\text{cat}(\mathcal{Z})$  critical points near  $\mathcal{Z}$ .*

In the above statement, we have denoted by  $\text{cat}(\mathcal{Z})$  the Lusternik–Schnirelmann category of the topological space  $\mathcal{Z}$  with respect to itself, namely the least integer  $k$  such that  $\mathcal{Z}$  can be covered by  $k$  closed subsets of  $\mathcal{Z}$  which are contractible to a point in  $\mathcal{Z}$ .

We will apply Theorem 4 separately for the clockwise and for the counter-clockwise rotating solutions. Let us deal with those last ones, taking as  $\mathcal{Z}$  the manifold of circular solutions of radius  $\rho_*$  of the unperturbed system  $(S_0)$  rotating counter-clockwise. (The other case is completely analogous.) Clearly enough,  $\mathcal{Z} \subseteq \mathcal{E}_*$ . Once we choose  $x_*(t)$ , one of such circular solutions,  $\mathcal{Z}$  can be parametrized as a curve  $\gamma : [0, 2\pi] \rightarrow \mathcal{E}_*$ , e.g.,

$$\gamma(\theta)(t) = x_*(t + \theta).$$

Hence,  $\mathcal{Z}$  is homeomorphic to  $S^1$ , and  $\text{cat}(\mathcal{Z}) = 2$ . Let us now verify the assumptions of Theorem 4.

Condition (i) holds since every  $x \in \mathcal{Z}$  is a solution of  $(S_0)$ , and hence  $\varphi'_0(x) = 0$ .

In order to prove condition (ii) notice that, taking  $v, \xi \in \mathcal{E}$ , one has  $\varphi''_0(x)v = \xi$  if and only if

$$\int_0^{m\ell T} \left[ \langle \dot{v} - \dot{\xi}, \dot{w} \rangle - \left\langle \frac{g'(|x|)}{|x|} \langle x, v \rangle x + g(|x|)v + \xi, w \right\rangle \right] dt = 0,$$

for every  $w \in \mathcal{E}$ . Setting  $u = \xi - v$ , with the same argument used to prove Lemma 3 we can then deduce that  $u$  is twice continuously differentiable, and satisfies the differential equation

$$\ddot{u} - u = \frac{g'(|x|)}{|x|} \langle x, v \rangle x + (g(|x|) + 1)v. \quad (9)$$

By the Ascoli–Arzelà Theorem it can be proved that the linear operator  $P : \mathcal{E} \rightarrow \mathcal{E}$ , taking every  $v \in \mathcal{E}$  into the unique solution  $u \in \mathcal{E}$  of (9), is a compact operator. Being  $\varphi''_0(x) = \text{Id} + P$ , the Fredholm Alternative Theorem guarantees that  $\varphi''_0(x)$  has a finite dimensional kernel and a closed image. Since it is selfadjoint,  $\ker \varphi''_0(x) = [\text{Im } \varphi''_0(x)]^\perp$ , and we conclude that  $\varphi''_0(x)$  is a Fredholm operator of index zero.

Finally, in order to verify (iii), we have to prove that  $T_x \mathcal{Z} \supseteq \ker \varphi''_0(x)$  for every  $x \in \mathcal{Z}$  (the other inclusion is always true because  $x$  is a critical point). So, we choose  $x \in \mathcal{Z}$  and  $v \in \ker \varphi''_0(x)$ . Without loss of generality, we assume that  $x(0) = (\rho_*, 0)$  so that, by (3), using the complex notation,

$$x(t) = \rho_* e^{\frac{2\pi i}{\tau_*} t} = \rho_* e^{\frac{2\pi n i}{m\ell T} t}.$$

We want to prove that  $v \in T_x \mathcal{Z}$ . Taking  $\theta \in \mathbb{R}$  such that  $x(t) = x_*(t + \theta)$ , i.e.,  $x = \gamma(\theta)$ , we notice that

$$T_x \mathcal{Z} = \{\mu \gamma'(\theta) : \mu \in \mathbb{R}\} = \{i\nu x : \nu \in \mathbb{R}\}.$$

Particularizing equation (9) with  $\xi = 0$ , we see that any  $v \in \ker \varphi_0''(x)$  satisfies the differential equation

$$-\ddot{v} = g(|x|)v + \frac{g'(|x|)}{|x|} \langle v, x \rangle x. \quad (10)$$

Here, as well, using the complex notation we write  $v : \mathbb{R} \rightarrow \mathbb{C}$ , and since we look for  $m\ell T$ -periodic solutions, we have the corresponding Fourier series

$$v(t) \sim \sum_{k \in \mathbb{Z}} c_k e^{\frac{2\pi k i}{m\ell T} t},$$

with  $c_k \in \mathbb{C}$ . Moreover, since  $v(t + \ell T) = \mathcal{R}^n v(t)$ , it has to be

$$\sum_{k \in \mathbb{Z}} c_k e^{\frac{2\pi k i}{m\ell T} (t + \ell T)} = e^{\frac{2\pi n i}{m}} \sum_{k \in \mathbb{Z}} c_k e^{\frac{2\pi k i}{m\ell T} t},$$

so that

$$c_k e^{\frac{2\pi k i}{m}} = c_k e^{\frac{2\pi n i}{m}}, \quad \text{for every } k \in \mathbb{Z},$$

hence

$$\text{either } c_k = 0, \text{ or } k \in n + m\mathbb{Z}. \quad (11)$$

Now, we write equation (10) as

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \left( \frac{2\pi k}{m\ell T} \right)^2 c_k e^{\frac{2\pi k i}{m\ell T} t} &= g(\rho_*) \sum_{k \in \mathbb{Z}} c_k e^{\frac{2\pi k i}{m\ell T} t} + \\ &+ g'(\rho_*) \left\langle \sum_{k \in \mathbb{Z}} c_k e^{\frac{2\pi k i}{m\ell T} t}, \rho_* e^{\frac{2\pi n i}{m\ell T} t} \right\rangle e^{\frac{2\pi n i}{m\ell T} t}. \end{aligned}$$

Recalling that, by (3),

$$g(\rho_*) = \left( \frac{2\pi}{\tau_*} \right)^2 = \left( \frac{2\pi n}{m\ell T} \right)^2,$$

and setting

$$\alpha = \frac{g'(\rho_*) \rho_*}{2} \left( \frac{m\ell T}{2\pi} \right)^2,$$

we get

$$\sum_{k \in \mathbb{Z}} (k^2 - n^2) c_k e^{\frac{2\pi k i}{m\ell T} t} = 2\alpha \left\langle \sum_{k \in \mathbb{Z}} c_k e^{\frac{2\pi k i}{m\ell T} t}, e^{\frac{2\pi n i}{m\ell T} t} \right\rangle e^{\frac{2\pi n i}{m\ell T} t}.$$



Remark that, having assumed  $g'(\rho_*) \neq 0$ , it follows that  $\alpha \neq 0$ , as well. Since  $\langle z_1, z_2 \rangle = \operatorname{Re}(z_1 z_2^*)$  and  $\operatorname{Re}(z) = \frac{1}{2}(z + z^*)$ , we have

$$\begin{aligned} \sum_{k \in \mathbb{Z}} (k^2 - n^2) c_k e^{\frac{2\pi k i}{m\ell T} t} &= 2\alpha \operatorname{Re} \left( \sum_{k \in \mathbb{Z}} c_k e^{\frac{2\pi(k-n)i}{m\ell T} t} \right) e^{\frac{2\pi n i}{m\ell T} t} \\ &= \alpha \sum_{k \in \mathbb{Z}} \left[ c_k e^{\frac{2\pi k i}{m\ell T} t} + c_k^* e^{\frac{2\pi(2n-k)i}{m\ell T} t} \right]. \end{aligned}$$

We then get

$$\sum_{k \in \mathbb{Z}} (k^2 - n^2 - \alpha) c_k e^{\frac{2\pi k i}{m\ell T} t} = \alpha \sum_{k \in \mathbb{Z}} c_k^* e^{\frac{2\pi(2n-k)i}{m\ell T} t},$$

which, after a shift in the indices, becomes

$$\sum_{k \in \mathbb{Z}} (k^2 - n^2 - \alpha) c_k e^{\frac{2\pi k i}{m\ell T} t} = \alpha \sum_{k \in \mathbb{Z}} c_{2n-k}^* e^{\frac{2\pi k i}{m\ell T} t}.$$

Hence, we must have that

$$(k^2 - n^2 - \alpha) c_k = \alpha c_{2n-k}^*, \quad \text{for every } k \in \mathbb{Z}.$$

Taking  $k = n$ , we immediately observe that  $c_n = -c_n^*$ , i.e.,  $\operatorname{Re}(c_n) = 0$ . On the other hand, by a change of indices, we have that

$$((n+j)^2 - n^2 - \alpha) c_{n+j} = \alpha c_{n-j}^*, \quad \text{for every } j \in \mathbb{Z}. \quad (12)$$

So, also

$$((n-j)^2 - n^2 - \alpha) c_{n-j} = \alpha c_{n+j}^*, \quad \text{for every } j \in \mathbb{Z},$$

which is equivalent to

$$\alpha c_{n+j} = ((n-j)^2 - n^2 - \alpha) c_{n-j}^*, \quad \text{for every } j \in \mathbb{Z}. \quad (13)$$

We now solve the system (12)–(13). Since  $\alpha \neq 0$ , substitution yields

$$(j^2 - \alpha + 2nj) (j^2 - \alpha - 2nj) c_{n-j}^* = \alpha^2 c_{n-j}^*,$$

hence

$$\left[ (j^2 - \alpha)^2 - 4n^2 j^2 - \alpha^2 \right] c_{n-j}^* = 0,$$

and finally

$$j^2 [j^2 - 2\alpha - 4n^2] c_{n-j}^* = 0, \quad \text{for every } j \in \mathbb{Z}. \quad (14)$$

From (11) we know that  $c_{n-j} = 0$  when  $j \notin m\mathbb{Z}$ . Let now  $j = mp$ , for some  $p \in (\mathbb{Z} \setminus \{0\})$ ; since

$$2\alpha + 4n^2 = \left(\frac{m\ell T}{2\pi}\right)^2 (4g(\rho_*) + g'(\rho_*)\rho_*) = \left(\frac{m\ell T}{2\pi}\right)^2 h(\rho_*),$$

from (14) we deduce that

$$\left[p^2 - \left(\frac{\ell T}{2\pi}\right)^2 h(\rho_*)\right] c_{n-mp} = 0,$$

and assumption (5) leads to  $c_{n-mp} = 0$ . We then conclude that

$$c_k = 0, \quad \text{for every } k \neq n.$$

Therefore we have proved that, if  $v \in \ker \varphi_0''(x)$ , it has the form

$$v(t) = c_n e^{\frac{2\pi i n}{m\ell T} t},$$

where  $c_n$  is a pure imaginary complex number, hence it can be written as  $c_n = i\sigma$  with  $\sigma \in \mathbb{R}$ . Therefore,

$$v(t) = i\sigma e^{\frac{2\pi i n}{m\ell T} t} = i \frac{\sigma}{\rho_*} x(t).$$

Hence,  $v \in T_x \mathcal{Z}$ , and this concludes the proof that (iii) holds true.

Theorem 1 is thus completely proved.

### 3 A myriad of periodic solutions

As a variant of system  $(K_\varepsilon)$ , we may consider the generalized Kepler problem

$$\ddot{x} + \frac{c}{|x|^\gamma} x = \varepsilon \nabla_x V(t, x), \quad (\tilde{K}_\varepsilon)$$

where  $c$  is a positive constant, and  $\gamma \neq 0$ . We assume again that (1) holds, where  $\mathcal{R}$  is a rotation around the origin of angle  $2\pi/m$ . We fix a circular orbit  $x_*(t)$  of the unperturbed system  $(\tilde{K}_0)$ , with radius  $\rho_*$  and minimal period  $\tau_*$ , and we assume (3), for some positive integers  $\ell, n$ . In this case, the function  $h(\rho)$  coincides with  $(4 - \gamma)g(\rho) = (4 - \gamma)c\rho^{-\gamma}$ , and we have the following immediate consequence of Theorem 1.

**Corollary 5.** *If (3) is verified, and*

$$(4 - \gamma) \left( \frac{\ell T}{\tau_*} \right)^2 \notin \{k^2 : k \in \mathbb{N} \setminus \{0\}\}, \quad (15)$$

*then the same conclusion of Theorem 1 holds for system  $(\tilde{K}_\varepsilon)$ .*

Notice that Corollary 2 is a direct consequence of Corollary 5. However, condition (15) could be verified even if  $m = 1$ , in which case no symmetry is required on the perturbing term. This surely happens, e.g., if  $\gamma \geq 4$  or, assuming  $\gamma < 4$  and recalling (3), when  $\sqrt{4 - \gamma}$  is an irrational number.

One could ask whether a condition like (15) is really needed. Indeed, such a condition is not found in the paper by Ambroseti and Coti Zelati [1]. However, there is an unclear point in their proof, when they deduce (1.14)<sub>k</sub> from (1.13), claiming that  $n \pm n(4 - \gamma)^{1/2}$  cannot be an integer when  $\gamma \neq 3$ . It could be possible that a weaker nondegenerate condition would guarantee the existence of periodic solutions, but we will not investigate further this point.

Nevertheless, we will now prove the following rather surprising result.

**Theorem 6.** *Assume  $\gamma > 0$  and, whenever  $\gamma = 3$ , then  $m \geq 2$ . For any circular solution  $x_*(t)$  of the unperturbed system  $(\tilde{K}_0)$ , for any positive integer  $N$  and any  $\sigma > 0$ , there exists  $\bar{\varepsilon} > 0$  such that, if  $|\varepsilon| \leq \bar{\varepsilon}$ , equation  $(\tilde{K}_\varepsilon)$  has at least  $4N$  periodic solutions  $x(t)$ , whose orbits are near the one of  $x_*(t)$ , in the sense of (6) or (7).*

Proof. Making  $\rho$  vary near  $\rho_*$ , we find infinitely many of its values for which  $mT/\tau(\rho)$  is a rational number, i.e.,

$$\frac{mT}{\tau(\rho)} = \frac{n(\rho)}{\ell(\rho)}, \quad (16)$$

for some positive integers  $\ell(\rho)$ ,  $n(\rho)$ . We would like to apply Corollary 5 when  $\rho_*$  and  $\tau_*$  are replaced by  $\rho$ ,  $\tau(\rho)$ , respectively. To this aim, we need to prove that condition (15) is satisfied, i.e., that

$$(4 - \gamma) \left( \frac{\ell(\rho)T}{\tau(\rho)} \right)^2 \notin \{k^2 : k \in \mathbb{N} \setminus \{0\}\}, \quad (17)$$

for several values of  $\rho$  near  $\rho_*$ .

If  $\gamma \geq 4$ , then condition (17) is surely satisfied. Assume then  $\gamma < 4$ , and consider the number  $\kappa = \sqrt{4 - \gamma}/m$ . Then, by (16), condition (17) is equivalent to

$$\kappa \frac{\ell(\rho)mT}{\tau(\rho)} = \kappa n(\rho) \notin \mathbb{N}. \quad (18)$$

If  $\kappa$  is not rational, condition (18) is surely satisfied. Otherwise, let  $\kappa = p/q \in \mathbb{Q}$ , for some positive integers  $p, q$ . We claim that  $q$  cannot be equal to 1. Indeed, if it were  $q = 1$ , i.e.,  $\gamma = 4 - (mp)^2$ , since  $m, p$  are positive integers, we would have two possibilities: if  $m = p = 1$ , then  $\gamma = 3$ , in which case we contradict the assumption that  $m \geq 2$ ; on the other hand, if  $m \neq 1$  or  $p \neq 1$ , then  $\gamma \leq 0$ , contradicting the assumption that  $\gamma > 0$ .

Now, among all the fractions in (16) we can select only those for which  $n(\rho)$  is relatively prime with  $q$ . There are infinitely many of them, and

$$\kappa n(\rho) = p \frac{n(\rho)}{q} \notin \mathbb{N}.$$

We have then proved that condition (18) is verified by infinitely many values of  $\rho$  near  $\rho_*$ , so that the assumptions of Corollary 5 are satisfied.

Take  $N$  of these values of  $\rho$  satisfying the conditions (16) and (17) above, denote them by  $\rho_1, \dots, \rho_N$ , and let  $x_*^1(t), \dots, x_*^N(t)$  be some circular solutions of  $(\tilde{K}_0)$  having those radii. By Corollary 5, there are  $N$  positive numbers  $\bar{\varepsilon}_1, \dots, \bar{\varepsilon}_N$  such that, if  $|\varepsilon| \leq \bar{\varepsilon}_j$ , equation  $(\tilde{K}_\varepsilon)$  has at least four periodic solutions  $x_j(t)$  whose orbit is near the one of  $x_*^j(t)$ . If  $|\varepsilon|$  is small enough, all these solutions must be different from one another, thus proving the result.  $\square$

As an illustrative example of application of Corollary 5 and Theorem 6, we consider the motion of a body having zero mass under the gravitational attraction of  $m + 1$  bodies, in the following configuration. The first of them is fixed at the origin, and has a mass equal to  $M_0$ . The other  $m$  bodies move along the lines passing through the origin with angular directions  $2\pi j/m$ , with  $j = 1, \dots, m$ , and have small masses equal to the same value  $\varepsilon$ . We assume that these  $m$  small masses move symmetrically and periodically in time (see Figure 1).

Precisely, let  $p : \mathbb{R} \rightarrow [0, +\infty[$  be a  $T$ -periodic function, and denote by  $P : \mathbb{R} \rightarrow \mathbb{R}^2$  the vector-valued function  $P(t) = (p(t), 0)$ . The equation of motion for our zero-mass body will then be of the type  $(K_\varepsilon)$ , with  $c = GM_0$  (here,  $G$  is the universal gravitational constant), and

$$V(t, x) = \sum_{j=1}^m \frac{G}{|x - \mathcal{R}^j P(t)|}.$$

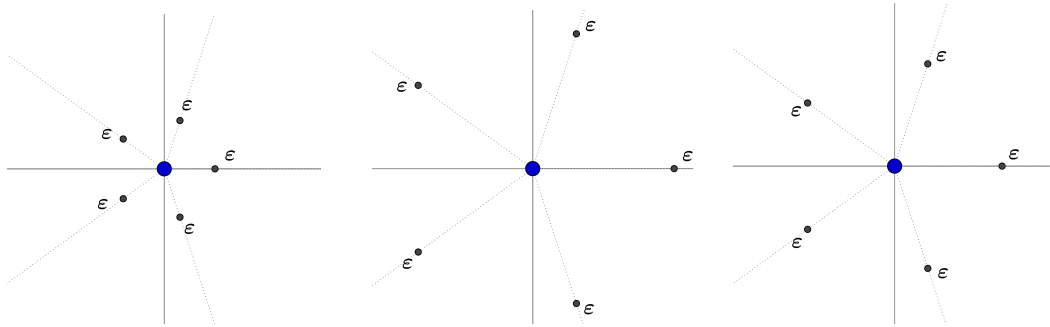


Figure 1: The case  $m = 5$ , at three different times.

As usual,  $\mathcal{R}$  denotes the rotation of angle  $2\pi/m$  about the origin. It is easily seen that (1) is satisfied, so that Corollary 5 (or Corollary 2) and Theorem 6 can be applied, providing the existence of plenty of periodic orbits.

**Acknowledgement.** This work has been partially supported by the Istituto Nazionale di Alta Matematica (INdAM).

## References

- [1] A. Ambrosetti and V. Coti Zelati, Perturbation of Hamiltonian systems with Keplerian potentials, *Math. Z.* 201 (1989), 227–242.
- [2] A. Ambrosetti and V. Coti Zelati, *Periodic Solutions of Singular Lagrangian Systems*, Birkhäuser, Basel, 1993.
- [3] A. Ambrosetti, V. Coti Zelati and I. Ekeland, Symmetry breaking in Hamiltonian systems, *J. Differential Equations* 67 (1987), 165–184.
- [4] A. Boscaggin and R. Ortega, Periodic solutions of a perturbed Kepler problem in the plane: from existence to stability, *J. Differential Equations* 261 (2016), 2528–2551.
- [5] H. Cabral and C. Vidal, Periodic solutions of symmetric perturbations of the Kepler problem, *J. Differential Equations* 163 (2000), 76–88.
- [6] A. Capozzi, S. Solimini and S. Terracini, On a class of dynamical systems with singular potential, *Nonlinear Anal.* 16 (1991), 805–815.
- [7] D. Chillingworth, Bifurcation from an orbit of symmetry, in “Singularities and Dynamical Systems” (S. N. Pnevmticos, Ed.), North-Holland, Amsterdam, 1985.

- [8] B. Cordani, *The Kepler Problem. Group Theoretical Aspects, Regularization and Quantization, with Application to the Study of Perturbations*, Birkhäuser, Basel, 2003.
- [9] M. Degiovanni and F. Giannoni, *Nonautonomous perturbations of Newtonian potentials*, *Nonlinear Variational Problems, Vol. II (Isola d'Elba, 1986)*, 187–195, *Pitman Res. Notes Math. Ser.*, 193, Longman, Harlow, 1989.
- [10] A. Fonda and A.C. Gallo, *Radial periodic perturbations of the Kepler problem*, *Celest. Mech. Dyn. Astr.* 128 (2017), 257–268.
- [11] A. Fonda and R. Toader, *Periodic orbits of radially symmetric Keplerian-like systems: A topological degree approach*, *J. Differential Equations* 244 (2008), 3235–3264.
- [12] A. Fonda and R. Toader, *Periodic solutions of radially symmetric perturbations of Newtonian systems*, *Proc. Amer. Math. Soc.* 140 (2012), 1331–1341.
- [13] A. Fonda, R. Toader and P.J. Torres, *Periodic motions in a gravitational central field with a rotating external force*, *Celestial Mech. Dynam. Astronom.* 113 (2012), 335–342.
- [14] A. Fonda and A.J. Ureña, *Periodic, subharmonic, and quasi-periodic oscillations under the action of a central force*, *Discrete Contin. Dyn. Syst.* 29 (2011), 169–192.
- [15] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Springer, New York, 1989.
- [16] M. Reeken, *Stability of critical points under small perturbations. II Analytic theory*, *Manuscripta Math.* 8 (1973), 69–92.
- [17] E. Serra and S. Terracini, *Noncollision solutions to some singular minimization problems with Keplerian-like potentials*, *Nonlinear Anal.* 22 (1994), 45–62.
- [18] A. Vanderbauwhede, *Local Bifurcation and Symmetry*, *Research Notes in Math.*, Pitman, New York, 1982.
- [19] C. Vidal, *Periodic solutions for any planar symmetric perturbation of the Kepler problem*, *Celestial Mech. Dynam. Astronom.* 80 (2001), 119–132.

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Mathematics Subject Classification: 34C25

Keywords: Kepler problem; periodic solutions; rotational symmetric systems