FANO CONGRUENCES OF INDEX 3 AND ALTERNATING 3-FORMS

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Abstract. We study congruences of lines X_ω defined by a sufficiently general choice of an alternating 3-form ω in $n+1$ dimensions, as Fano manifolds of index 3 and dimension $n-1$. These congruences include the G_2-variety for $n=6$ and the variety of reductions of projected $\mathbb{P}^2 \times \mathbb{P}^2$ for $n=7$.

We compute the degree of X_ω as the n-th Fine number and study the Hilbert scheme of these congruences proving that the choice of ω bijectively corresponds to X_ω except when $n=5$. The fundamental locus of the congruence is also studied together with its singular locus: these varieties include the Coble cubic for $n=8$ and the Peskine variety for $n=9$.

The residual congruence Y of X_ω with respect to a general linear congruence containing X_ω is analysed in terms of the quadrics containing the linear span of X_ω. We prove that Y is Cohen-Macaulay but non-Gorenstein in codimension 4. We also examine the fundamental locus G of Y of which we determine the singularities and the irreducible components.

Contents

1. Introduction

Let V be an $(n+1)$-dimensional $\mathbb{K}$-vector space, and let $\mathbb{G} := G(2,V) \subset \mathbb{P}(\bigwedge^2 V)$ be the Grassmannian of 2-dimensional $\mathbb{K}$-vector subspaces of V, or, equivalently, the Grassmannian of lines in $\mathbb{P}^n = \mathbb{P}(V)$. A *congruence of lines* in the projective space $\mathbb{P}(V)$ is an $(n-1)$-dimensional closed subvariety of $\mathbb{G}$. A *linear congruence* is a congruence formed by the proper intersection of $\mathbb{G}$ with a linear space of codimension $n-1$ in $\mathbb{P}(\bigwedge^2 V)$. In this paper we consider congruences of lines that are proper components of linear congruences. A general 3-form $\omega \in \bigwedge^3 V^*$ defines a linear subspace $\Lambda_\omega := \{[L] \in \mathbb{P}(\bigwedge^2 V) \mid \omega(L) = 0\}$

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of codimension $n+1$ in $\mathbb{P}(\bigwedge^2 V)$. The intersection $X_\omega := \mathbb{G} \cap \Lambda_\omega$ is a congruence of lines, the first main object of this paper. A general 2-dimensional subspace $\langle x, y\rangle \subset V^*$ defines a codimension $n-1$ linear subspace $\Lambda_\omega^{xy} \subset \mathbb{P}(\bigwedge^2 V)$, that contains Λ_ω. The intersection

$$\Lambda_\omega^{xy} \cap \mathbb{G} = X_\omega \cup Y_{\omega,x\wedge y}$$

is a reducible linear congruence, with one component X_ω and the other component a congruence $Y_{\omega,x\wedge y}$. The congruence $Y_{\omega,x\wedge y}$—called the *residual congruence*, and sometimes denoted by Y for simplicity—is the second main object of this paper.

The *order* of a congruence is the number of lines in the congruence that pass through a general point in $\mathbb{P}(V)$. More generally, the *multidegree* of a congruence is the sequence of the coefficients in the expression of the congruence as a linear combination of Schubert cycles in the Chow ring of $\mathbb{G}$: the *i-th multidegree* $i = 0, \dots, \left[\frac{n-1}{2}\right]$, is the number of lines contained in a general $\mathbb{P}^{n-i} \subset \mathbb{P}^n$ that intersect a general $\mathbb{P}^i$ contained in $\mathbb{P}^{n-i}$, with $i < n-i$. A linear congruence has order one. Since X_ω and $Y_{\omega,x\wedge y}$ are components of a linear congruence, one of them has order one, the other has order zero. The order is the first component of the multidegree referred to above (i.e. the 0-th multidegree), so X_ω has order one when n is even, while $Y_{\omega,x\wedge y}$ has order one when n is odd.

The *fundamental locus* of a congruence is the locus of points in $\mathbb{P}(V)$ through which there are infinitely many lines of the congruence.

Congruences of lines of order one appear naturally in several interesting problems in geometry, and thus motivated this study. These include the classification of varieties with one apparent double point ([CMR04], [CR11]), the degree of irrationality of general hypersurfaces ([BCDeP14], [BDeP15]) and in hyperbolic conservation laws, so called Temple systems of partial differential equations ([AF01], [DePM05])). For a survey of order one congruences of lines, see [DePM07].

A well-known fact is that while the general linear congruence in $\mathbb{G}$ is a Fano variety of index 2, the congruence X_ω is a Fano variety of index 3 (Theorem 3.9). Varieties X_ω for small n have been studied by many authors, both by the construction from a 3-form as above, and by other constructions: When $n = 3, 4, 5, 6$ and ω is general, then X_ω is a plane, a quadric threefold, the Segre product $\mathbb{P}^2 \times \mathbb{P}^2$ and the closed orbit of the Lie group G_2 in its adjoint representation, respectively. For the next values of n there are more recent studies by Peskine in [Pes15] in the cases $n = 7, 9$, by Gruson and Sam in [GS15] in the case $n = 8$ and by Han in [Han15] in the case $n = 9$.

In this paper we present and prove general properties, some well-known and some new, of the congruences X_ω and $Y_{\omega,x\wedge y}$, for sufficiently general ω, x and y. After presenting equations 3.6, and locally free resolutions of their ideals (Theorems 3.9 and 7.2), we give the multidegree of these congruences in the Chow ring of the Grassmannian (Propositions 3.12 and 7.8).

Both X_ω and $Y_{\omega,x\wedge y}$ are improper intersections of the Grassmannian with a linear space: $Y_{\omega,x\wedge y}$ is contained in a codimension n linear space which we shall denote by $\Lambda_{\omega,x\wedge y}$ (i.e. $\Lambda_{\omega,x\wedge y}$ is a hyperplane in Λ_ω^{xy}), see Definition (2.10), Remark 2.12 and Proposition 7.5. We give different characterisations of their linear spans. In particular we identify the quadrics in the ideal of $\mathbb{G}$—called also *Plücker quadrics*—that contain the linear span of X_ω (Proposition 6.3). These quadrics necessarily have a large linear singular locus. We show (Theorem 6.7) that this singular locus is the linear span of a congruence $X'_{\omega'}$ of dimension one less.

We study the fundamental loci of X_ω and of its residual congruences, giving equations and numerical invariants of the fundamental locus F_ω of X_ω (Proposition 4.4) and numerical invariants for the fundamental locus $G_{\omega,x\wedge y}$ of $Y_{\omega,x\wedge y}$ (Theorem 7.15).

Peskine showed recently that if an irreducible congruence of lines is Cohen-Macaulay and has order one, then the lines of the congruence are the k-secant lines to its fundamental locus for some k [Pes15, Theorem 3.2]. The integer k is called the *secant index* of the congruence.

The congruences X_ω and $Y_{\omega,x\wedge y}$ are both Cohen-Macaulay (Theorem 3.8 and Proposition 7.2, (3)). In fact X_ω is smooth, while $Y_{\omega,x\wedge y}$ is singular and not even Gorenstein when $n > 5$ (Proposition 7.12). Moreover, we prove that both X_ω and $Y_{\omega,x\wedge y}$ are arithmetically Cohen-Macaulay (Corollary 3.10 and Proposition 7.10, (1)). Since X_ω is also subcanonical, it is arithmetically Gorenstein also (Corollary 3.10).

When n is odd, then X_ω has order one, the fundamental locus $F_\omega \subset \mathbb{P}(V)$ has codimension 3 and is singular in codimension 10. The secant index of X_ω is $(n-1)/2$. The congruence $Y_{\omega,x\wedge y}$ has order zero and the fundamental locus $G_{\omega,x\wedge y}$ is a hypersurface of degree $(n-1)/2$ that contains F_ω.

When n is even, then X_ω has order zero, and the fundamental locus F_ω is a hypersurface of degree $(n-2)/2$, while $Y_{\omega,x\wedge y}$ has order one and the fundamental locus $G_{\omega,x\wedge y} = \Pi \cup G_0$ is the union of the codimension 2 linear space $\Pi = \{x = y = 0\} \subset \mathbb{P}(V)$ and a codimension 3 subvariety G_0 contained in F_ω (Theorem 7.15). The secant index of $Y_{\omega,x\wedge y}$ is $n/2$, and each line in the congruence is $(n-2)/2$-secant to G_0 and intersects Π (Theorem 7.17).

The quadrics in the ideal of $\mathbb{G}$ naturally correspond to elements of $\bigwedge^4 V^*$, and the quadrics in this ideal that contain the linear span Λ_ω are naturally characterised via this correspondence. Moreover, for these quadrics, their singular locus is the linear span of a congruence of the same type and dimension one less (Theorem 6.7).

Therefore we introduce and present basic results on this correspondence in Section 2. The congruence X_ω is defined and introduced with basic properties in Section 3. Section 4 is devoted to the fundamental locus of F_ω of X_ω, while properties of the Hilbert scheme of X_ω are studied in Section 5. In Section 6 we identify the quadrics in the ideal of $\mathbb{G}$ that contain the linear span of X_ω. A general linear space of maximal dimension in such a quadric contains a congruence X'_ω or a congruence $Y_{\omega',x\wedge y}$ for some 3-form ω' and some linear forms $x, y \in V^*$. The final section 7 is devoted to the congruence $Y_{\omega,x\wedge y}$ and its fundamental locus. Some of the main properties and invariants of the congruences X_ω and Y and their fundamental loci are collected in Tables 1 and 2 at the end of the paper.

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1.1. **Notation.** We shall work over an algebraically closed field $\mathbb{K}$ of characteristic zero.

Throughout the paper, V will be an $(n+1)$-dimensional $\mathbb{K}$-vector space, and $\mathbb{G} := G(2, V)$ will be the Grassmannian of 2-dimensional vector subspaces of V.

First, we fix a basis $(e_0, \dots, e_n)$ for V. Let $(x_0, \dots, x_n)$ be the dual basis in V^*. Then, the n-dimensional projective space defined by the lines of V is $\mathbb{P}(V) = \mathrm{Proj}(\mathbb{K}[x_0 \dots, x_n]) = \mathrm{Proj}(\mathrm{Sym}(V^*))$. We shall denote—as it is the custom—by $\mathcal{T}_{\mathbb{P}(V)}$ and $\Omega^1_{\mathbb{P}(V)}$ the tangent and cotangent bundles of $\mathbb{P}(V)$, respectively. Moreover, as usual, $\Omega^k_{\mathbb{P}(V)} := \bigwedge^k \Omega^1_{\mathbb{P}(V)}$, $\mathcal{T}_{\mathbb{P}(V)}(h) := \mathcal{T}_{\mathbb{P}(V)} \otimes \mathcal{O}_{\mathbb{P}(V)}(h)$, etc.

We will adopt the following convention on parenthesis: $S^m\mathcal{E}(t)$ means the twist by $\mathcal{O}(t)$ of the m-th symmetric product of $\mathcal{E}$, and similarly for exterior powers and any other Schur functor of $\mathcal{E}$.

We consider the Grassmannian $\mathbb{G}$ with its Plücker embedding: $\mathbb{G} \subset \mathbb{P}(\bigwedge^2 V)$, and we fix Plücker coordinates $p_{i,j}$ on it: for example the point defined by $p_{i,j} = 0$ if $(i,j) \neq (0,1)$, i.e. the point $[1, 0, \dots, 0] \in \mathbb{G}$, corresponds to the subspace generated by e_0 and e_1.

Since there is no standard notation about universal and quotient bundles on Grassmannians, we fix it in the following. On $\mathbb{G}$ we denote the *universal subbundle* of rank 2 by $\mathcal{U}$ and the *quotient bundle* of rank $n-1$ by $\mathcal{Q}$. They fit in the following exact sequence

$$0 \to \mathcal{U} \to V \otimes \mathcal{O}_{\mathbb{G}} \to \mathcal{Q} \to 0, \tag{1.1}$$

where the universal subbundle $\mathcal{U}$ has as its fibre over $\ell \in \mathbb{G}$ the 2-dimensional vector subspace L of V which corresponds to the point ℓ, while the quotient bundle $\mathcal{Q}$ has as its fibre over $\ell \in \mathbb{G}$ the quotient space V/L.

A *congruence of lines* in $\mathbb{P}(V)$ is a family of lines of dimension $n-1$. In other words, it is a closed subvariety—not necessarily irreducible—of dimension $n-1$ of $\mathbb{G}$. A *linear congruence* is a congruence corresponding to the proper intersection of $\mathbb{G}$ with a linear subspace of $\mathbb{P}(\bigwedge^2 V)$ of codimension $n-1$. Note that this definition of linear congruence is more restrictive than the one given by Peskine [Pes15], who considers linear but possibly improper sections of $\mathbb{G}$ of pure dimension $n-1$.

The *order* of a congruence Γ is the number of lines of Γ passing through a general point $P \in \mathbb{P}(V)$. In other words it is the degree of the intersection of Γ with the Schubert variety $\Sigma_P \subset \mathbb{G}$ of the lines passing through P.

Finally, for $x \in V^* \setminus \{0\}$ we denote by $V_x := \{x = 0\} \subset V$, the hyperplane of equation $x = 0$. Moreover, for two linearly independent forms $x, y \in V^*$, we denote by $V_{x\wedge y} := \{x = y = 0\} \subset V$ the corresponding codimension 2 subspace $V_{x\wedge y} = V_x \cap V_y$. As in the introduction, its projectivisation is $\Pi := \mathbb{P}(V_{x\wedge y}) \subset \mathbb{P}(V)$.

Let $\omega_x \in \bigwedge^3 V_x^*$ be the natural restriction of ω to V_x. If we choose a vector $e \in V$ such that $x(e) = 1$, then we have an identification $V_x^* \simeq e^\perp \subset V^*$, which induces an inclusion $\bigwedge^3 V_x^* \subset \bigwedge^3 V^*$. Therefore we get

$$\bigwedge^3 V^* = \bigwedge^3 V_x^* \oplus \bigwedge^2 V_x^* \wedge \langle x \rangle \tag{1.2}$$

which induces a unique decomposition $\omega = \omega_x + \beta_x \wedge x$.

We shall often decompose ω in this way, without specifying the choice of the vector e.

2. 3-forms, 4-forms and linear spaces in quadrics defining the Grassmannian of lines

In this section we discuss quadrics in $\mathbb{P}(\bigwedge^2 V)$ defined by 4-forms on V, and in particular by decomposable 4-forms $\eta \in \bigwedge^4 V^*$ that have a linear factor, i.e. $\eta = x \wedge \omega$ for some $x \in V^*$ and $\omega \in \bigwedge^3 V^*$. We shall study the rank of these quadrics. In the next sections we shall consider the intersection of maximal linear subspaces of such quadrics with the Grassmannian $\mathbb{G}$ of lines in $\mathbb{P}(V)$. In particular, interesting congruences of lines appear in such intersections.

We start by recalling the well-known isomorphism between the space of 4-forms on V and the space of quadratic forms in the ideal of $\mathbb{G}$.

Recall that the ideal of the Grassmannian $\mathbb{G} \subset \mathbb{P}(\bigwedge^2 V)$ is generated by the quadrics of rank 6 given by the Plücker relations. One way to obtain them is the following, see [Muk93]: let us take a general element $[L] \in \mathbb{P}(\bigwedge^2 V)$

$$[L] = [\sum_{0 \le i < j \le n} p_{i,j} e_i \wedge e_j];$$

then $[L] \in \mathbb{G}$ if and only if

$$L \wedge L = 0;$$

indeed

$$\begin{aligned}
[L \wedge L] &= [(\sum_{0 \le i < j \le n} p_{i,j} e_i \wedge e_j) \wedge (\sum_{0 \le h < k \le n} p_{h,k} e_h \wedge e_k)] \\
&= [(\sum_{0 \le i < j < h < k \le n} 2(p_{i,j}p_{h,k} - p_{i,h}p_{j,k} + p_{i,k}p_{j,h}) e_i \wedge e_j \wedge e_h \wedge e_k)]
\end{aligned}$$

and the vanishing of the coefficients gives the Plücker relations. More intrinsically, define the *reduced square* $L^{[2]} \in \bigwedge^4 V$ of a bivector $L = \sum_{0 \le i < j \le n} p_{i,j} e_i \wedge e_j \in \bigwedge^2 V$ as

$$L^{[2]} := \sum_{0 \le i<j<h<k \le n} \operatorname{Pfaff} \begin{pmatrix} 0 & p_{i,j} & p_{i,h} & p_{i,k} \\ -p_{i,j} & 0 & p_{j,h} & p_{j,k} \\ -p_{i,h} & -p_{j,h} & 0 & p_{h,k} \\ -p_{i,k} & -p_{j,k} & -p_{h,k} & 0 \end{pmatrix} e_i \wedge e_j \wedge e_h \wedge e_k.$$

Clearly we have $L \wedge L = 2L^{[2]}$ (i.e. this definition does not depend on the chosen basis of V), and $[L] \in \mathbb{G}$ if and only if $L^{[2]} = 0$.

Recall that one can extend the definition of quadratic form to define a *quadratic map*

$$q \colon V_1 \to V_2$$

as a map between $\mathbb{K}$-vector spaces V_1 and V_2 such that

$$q(aL) = a^2 q(L), \qquad \forall a \in \mathbb{K}, \forall L \in V_1$$

and that

$$B_q(L, L') := q(L + L') - q(L) - q(L')$$

is a bilinear map

$$B_q \colon V_1 \times V_1 \to V_2.$$

We then have the following

Proposition 2.1. *$\mathbb{G} \subset \mathbb{P}(\bigwedge^2 V)$ is scheme-theoretically the zero locus of the quadratic map associated to the reduced square*

$$q^{[2]} \colon \bigwedge^2 V \to \bigwedge^4 V$$
$$L \mapsto L^{[2]}.$$

In other words, the affine cone of the Grassmannian, $C(\mathbb{G})$, is the "kernel" (i.e. the inverse image of zero) of $q^{[2]}$:

$$0 \to C(\mathbb{G}) \cong \left(q^{[2]}\right)^{-1}(0) \hookrightarrow \bigwedge^2 V \xrightarrow{q^{[2]}} \bigwedge^4 V. \tag{2.1}$$

Therefore, giving an element of the vector space $I(\mathbb{G})_2$ of the quadratic forms in the ideal of Grassmannian $I(\mathbb{G})$, is equivalent to giving a 4-form in $\bigwedge^4 V^*$:

Corollary 2.2. *The $\mathrm{SL}(n+1)$-equivariant map*

$$\bigwedge^4 V^* \to \operatorname{Sym}^2 \left(\bigwedge^2 V^* \right)$$
$$\eta \mapsto q_\eta,$$

where q_η is the quadratic form on $\bigwedge^2 V$ defined by

$$q_\eta \colon \bigwedge^2 V \to \mathbb{K}$$
$$L \mapsto \eta(L^{[2]}),$$

is an isomorphism onto $I(\mathbb{G})_2 \subset \operatorname{Sym}^2 \left(\bigwedge^2 V^*\right)$.

Proof. We observe that q_η is obtained via composition from $q^{[2]}$: $\eta \mapsto q_\eta = \eta \circ q^{[2]}$; therefore the thesis follows from (2.1):

$$0 \to C(\mathbb{G}) \to \bigwedge^2 V \xrightarrow{q^{[2]}} \bigwedge^4 V \xrightarrow{\eta} \mathbb{K}.$$

□

Notation 2.3. We denote by Q_η the quadric defined by q_η in $\mathbb{P}(\bigwedge^2 V)$.

The singular locus of the quadric Q_η is defined by the kernel of the corresponding bilinear form. In terms of the corresponding 4-form η we get

$$\begin{aligned}\operatorname{Sing}(Q_\eta) &= \{[L] \in \mathbb{P}(\bigwedge^2 V) \mid \eta(L \wedge L') = 0, \forall L' \in \bigwedge^2 V\} \\ &= \{[L] \in \mathbb{P}(\bigwedge^2 V) \mid \rho_\eta(L) = 0\},\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}\rho_\eta\colon \bigwedge^2 V &\to \bigwedge^2 V^* \\ L &\mapsto \eta(L \wedge -)\end{aligned}$$

is defined by *contraction*. Note that ρ_η is the *polarity* associated to the quadric Q_η.

$Q_\eta \subset \mathbb{P}(\bigwedge^2 V)$ is a cone with vertex $\operatorname{Sing}(Q_\eta)$. Since $\bigwedge^2 V$ has dimension $\binom{n+1}{2}$ the rank of Q_η is equal to

$$\operatorname{rank}(q_\eta) = \binom{n+1}{2} - \dim \operatorname{Sing}(Q_\eta) - 1 = \binom{n+1}{2} - \dim \ker \rho_\eta = \operatorname{rank} \rho_\eta. \tag{2.3}$$

Let $\omega \in \bigwedge^i V^*$ be an i-form on V ($i \leq n+1$), then $\forall j \leq i$ the contraction defines two linear maps. The first one is

$$\begin{aligned}f_\omega\colon \bigwedge^j V &\to \bigwedge^{i-j} V^* \\ \alpha &\mapsto \omega(\alpha)\end{aligned}$$

and the other is its transpose (up to sign):

$$\begin{aligned}{}^t f_\omega\colon \bigwedge^{i-j} V &\to \bigwedge^j V^* \\ \beta &\mapsto \omega(\beta).\end{aligned}$$

Definition 2.4. For an i-form $\omega \in \bigwedge^i V^*$ we define its *j-rank* as the rank of f_ω (or, which is the same, as the rank of its transpose). If $j = i - 1$ we simply call it *rank* of ω.

Example 2.5. We shall be interested mainly in the following cases:

(1) For a 3-form $\omega \in \bigwedge^3 V^*$ its *rank* is its 2-rank. In other words

$$\operatorname{rank} \omega := \operatorname{rank}(\bigwedge^2 V \to V^* : L \mapsto \omega(L))$$

or, which is the same,

$$\operatorname{rank} \omega := \operatorname{rank}(V \to \bigwedge^2 V^* : e \mapsto \omega(e)).$$

(2) If $\beta \in \bigwedge^2 V^*$, then the *rank* of β is its 1-*rank*.

(3) If η is a 4-form, by (2.3) its 2-rank coincides with the rank of the quadric Q_η.

Remark 2.6. We note that Definition 2.4 of rank is different from the usual one for tensors, which is the minimum number of summands in an expression as sum of totally decomposable tensors.

In particular, in the case of a 2-form $\beta \in \bigwedge^2 V^*$, since f_β can be identified with the skew-symmetric matrix associated to β, the rank defined in Definition 2.4 is twice the usual rank of tensors, i.e. β is the sum of $\frac{1}{2} \operatorname{rank} \beta$ totally decomposable tensors.

We shall study now the rank of the quadratic form q_η, i.e. the rank of the linear map ρ_η, when η is a decomposable 4-form. We need some preparation.

Recall from Section 1.1 that when $x \in V^*$ is nonzero, then $V_x := \{x = 0\} \subset V$, denotes the subspace of equation $x = 0$. A 3-form $\omega \in \bigwedge^3 V^*$ has a decomposition $\omega = \omega_x + \beta_x \wedge x$, with $\beta_x \in \bigwedge^2 V_x^*$ (see 1.1).

Lemma 2.7. *Let $\eta \in \bigwedge^4 V^* \setminus \{0\}$.*

1. *If η is totally decomposable, i.e. $\eta = x \wedge x' \wedge x'' \wedge x'''$, where $x, x', x'', x''' \in V^*$ are linearly independent, then the 2-rank of η is $\operatorname{rank} \rho_\eta = 6$.*
2. *If $\eta = \beta \wedge x \wedge x' \neq 0$, and $\beta_{x,x'}$ is the restriction of the 2-form β to $V_{x\wedge x'} = V_x \cap V_{x'}$, then $\operatorname{rank} \rho_\eta = 2 \operatorname{rank} \beta_{x,x'} + 2 \leq 2n$.*
3. *If $\eta = \omega \wedge x$, where $x \in V^* \setminus \{0\}$ and ω_x is the restriction of ω to V_x, then $\operatorname{rank} \rho_\eta = 2 \operatorname{rank} \omega_x \leq 2n$.*

Proof. Case (1) is immediate, since the image of ρ_η is spanned by pairs of factors in η.

In Case (2), we may assume $x = x_0, x' = x_1$ and $\beta \in \bigwedge^2 \langle x_2, \dots, x_n \rangle$. For $L \in \bigwedge^2 V$, we write $L = L_{01} + v_0 \wedge e_0 + v_1 \wedge e_1 + c e_0 \wedge e_1$, where $L_{01} \in \bigwedge^2 \langle e_2, \dots, e_n \rangle$ and $v_0, v_1 \in \langle e_2, \dots, e_n \rangle$. Then

$$(\beta \wedge x_0 \wedge x_1)(L) = \beta(L_{01}) x_0 \wedge x_1 - \beta(v_0) x_1 + \beta(v_1) x_0 + c\beta,$$

so the formula follows.

In Case (3) we may assume $x = x_0$ and write $\omega \wedge x_0 = \omega_{x_0} \wedge x_0$. For $L \in \bigwedge^2 V$, we write $L = L_0 + v_0 \wedge e_0$, where $L_0 \in \bigwedge^2 \langle e_1, \dots, e_n \rangle$ and $v_0 \in \langle e_1, \dots, e_n \rangle$. Then

$$(\omega_{x_0} \wedge x_0)(L) = \omega_{x_0}(L_0) \wedge x_0 - \omega_{x_0}(v_0),$$

and the formula follows. ☐

The rank of the quadrics in the ideal of the Grassmannian $G(2, 6)$ has been studied by Mukai in [Muk93]. In this case $\bigwedge^4 V^* \cong \bigwedge^2 V^*$, so the three possible ranks of a 2-form give three possible ranks for quadrics in the ideal of $G(2, 6)$:

Proposition 2.8 ([Muk93, Proposition 1.4])**.** *Let $I_r \subset \mathbb{P}(I(G(2, 6))_2)$ be the set of quadrics of rank r in the ideal of $G(2, 6)$. Then $\mathbb{P}(I(G(2, 6))_2) = I_6 \cup I_{10} \cup I_{15}$, and $\dim I_6 = 8$, $\dim I_{10} = 13$, $\dim I_{15} = 14$. Moreover,*

- *If $[q] \in I_6$, then q is a Plücker quadric.*
- *If $[q] \in I_{10}$, then q is a linear combination of two Plücker quadrics.*
- *If $[q] \in I_{15}$, then $V(q)$ is smooth.*

2.1. **Genericity conditions on 3-forms.** In this paper our key objects are defined by "general" 3-forms. In this section, we introduce and discuss some of the relevant genericity conditions on $\omega \in \bigwedge^3 V^*$.

Definition 2.9 (Genericity conditions 1–3)**.** Let $\omega \in \bigwedge^3 V^*$. We will consider the following conditions:

(GC1): ω is indecomposable, i.e. the multiplication map

$$V^* \to \bigwedge^4 V^*$$
$$x \mapsto \omega \wedge x$$

is injective (the image of this map is the subspace $\omega \wedge V^* \subset \bigwedge^4 V^*$);

(GC2): ω has rank $n + 1$, i.e. the linear map

$$ {}^t f_\omega \colon \bigwedge^2 V \to V^* \qquad L \mapsto \omega(L) \tag{2.4}$$

is surjective, or, equivalently, the linear map

$$f_\omega\colon V \to \bigwedge^2 V^* \qquad v \mapsto \omega(v) \tag{2.5}$$

is injective;

(GC3): for any $v \in V$, $v \neq 0$, $\operatorname{rank} f_\omega(v) > 2$.

Remark 2.10. There are dependencies among these conditions:

(1) Condition (GC2) is satisfied, i.e. the rank of ω is $n+1$, only if $n \geq 4$. Indeed, when $n = 3$, any nonzero 3-form ω is totally decomposable so it has rank 3.
(2) Condition (GC1) is satisfied, i.e. $x \mapsto \omega \wedge x$ is injective, only if $n \geq 5$. Indeed, when $n \leq 4$, any 3-form is decomposable so the map $x \mapsto \omega \wedge x$ is not injective, see Example 4.10: with coordinates as in the example, x_0 is in the kernel.
(3) It is immediate to observe that condition (GC3) is more restrictive than both (GC1) and (GC2):

$$\text{(GC3)} \implies \text{(GC1)}, \ \text{(GC3)} \implies \text{(GC2)}.$$

(4) There are forms that satisfy (GC1) and not (GC2): Let $v_0 \in V$ be a nonzero vector and assume $\omega(v_0) = 0$, i.e. (GC2) is not satisfied. Then $\omega \in \bigwedge^3 V_0^*$, where $V_0^* = \{v^* \in V^* | v^*(v_0) = 0\}$. If the multiplication map by ω: $V_0^* \to \bigwedge^4 V_0^*$ is injective, then so is $V^* \to \bigwedge^4 V^*$ and (GC1) is satisfied.
(5) If n is even, there are forms that satisfy (GC2) and not (GC1): If ω does not satisfy (GC1), for a suitable choice of coordinates $\omega = x_0 \wedge \beta_0$. When n is even and β_0 is general, then $\operatorname{rank} \beta_0 = n$, hence $\operatorname{rank} \omega = n$ and condition (GC2) holds.
(6) If n is odd, (GC2) implies (GC1): If ω does not satisfy (GC1), for a suitable choice of coordinates $\omega = x_0 \wedge \beta_0$. When n is odd, $\operatorname{rank} \beta_0 \leq n-1$ so $\operatorname{rank} \omega \leq n-1$ and condition (GC2) does not hold.

Next proposition ensures that the conditions (GC1), (GC2), (GC3) are all satisfied by general 3-forms for n sufficiently large.

Proposition 2.11. *Let $\omega \in \bigwedge^3 V^*$ be general. If $n > 3$, then conditions (GC1) and (GC2) hold, and if $n > 5$, then $\omega(v)$ has rank at least 4 for every $v \in V$, i.e. condition (GC3) holds.*

Proof. To prove the first claim, it is enough to find an example for any n. For instance the following ones, depending on the congruence class of n modulo 3, will do the job: if $n+1 \equiv 0 \mod 3$ take $\omega = x_0 \wedge x_1 \wedge x_2 + \cdots + x_{n-2} \wedge x_{n-1} \wedge x_n$, if $n+1 \equiv 1 \mod 3$, $n > 3$, take $\omega = x_0 \wedge x_1 \wedge x_2 + \cdots + x_{n-3} \wedge x_{n-2} \wedge x_{n-1} + x_n \wedge x_0 \wedge x_3$, if $n+1 \equiv 2 \mod 3$, $n > 4$, take $\omega = x_0 \wedge x_1 \wedge x_2 + \cdots + x_{n-1} \wedge x_0 \wedge x_3 + x_n \wedge x_1 \wedge x_4$. To prove the second claim, we recall that the 2-forms of rank ≤ 2 describe a Grassmannian of dimension $2(n-1)$ in $\mathbb{P}(\bigwedge^2 V^*)$, hence for $n > 5$ the codimension is $> n+1$. □

2.2. **Quadrics $Q_{\omega\wedge x}$ and their linear subspaces.** Given a 3-form $\omega \in \bigwedge^3 V^*$, the 4-forms $\omega \wedge x$ for $x \in V^*$ define a linear space of quadrics

$$Q_{\omega\wedge x} \in I(\mathbb{G})_2, \qquad\qquad x \in V^*.$$

As we shall see, they are intimately related to the variety X_ω which is our main object of study.

In this section we identify linear subspaces of the quadrics $Q_{\omega\wedge x}$. If $\omega \in \bigwedge^3 V^*$ and $x, y \in V^*$, we define the following linear subspaces of $\mathbb{P}(\bigwedge^2 V)$:

$$\Lambda_\omega := \{[L] \in \mathbb{P}(\bigwedge^2 V) \mid \omega(L) = 0\}, \tag{2.6}$$

$$\Lambda_\omega^x := \{[L] \in \mathbb{P}(\bigwedge^2 V) \mid \omega(L) \wedge x = 0\}, \tag{2.7}$$

$$\Lambda_\omega^{xy} := \{[L] \in \mathbb{P}(\bigwedge^2 V) \mid \omega(L) \wedge x \wedge y = 0\}, \tag{2.8}$$

and moreover

$$\Lambda_{\omega_x} := \{[L] \in \mathbb{P}(\bigwedge^2 V) \mid x(L) = \omega(L) \wedge x = 0\}, \tag{2.9}$$

$$\Lambda_{\omega, x\wedge y} := \{[L] \in \mathbb{P}(\bigwedge^2 V) \mid (x \wedge y)(L) = \omega(L) \wedge x \wedge y = 0\}. \tag{2.10}$$

Furthermore we denote by $P_\omega := \Lambda_\omega^\perp \subset \mathbb{P}(\bigwedge^2 V)^*$ the subspace orthogonal to Λ_ω. Note the inclusions

$$\begin{array}{ccccc} \Lambda_\omega & \subset & \Lambda_\omega^x & \subset & \Lambda_\omega^{xy} \\ & & \cup & & \cup \\ & & \Lambda_{\omega_x} & & \Lambda_{\omega,x\wedge y} \end{array}$$

and the relation

$$\Lambda_{\omega_x} = \Lambda_\omega^x \cap \langle G(2, V_x)\rangle,$$

where V_x is the hyperplane $\{x = 0\}$ in V. Let us remark that the notation for Λ_{ω_x} is coherent with the one for Λ_ω, because ω_x denotes the restriction of ω to the hyperplane V_x. Moreover $\Lambda_{\omega,x\wedge y}$ is the intersection of Λ_ω^{xy} with the Schubert hyperplane generated by the planes in $\mathbb{G}$ meeting the codimension 2 subspace $V_{x\wedge y} := \{x = y = 0\}$.

Remark 2.12. The codimension of the spaces (2.6)-(2.10) depends on the rank of ω, and on the rank of ω_x as a form on V_x.

(1)
$$\Lambda_\omega = \mathbb{P}(\ker({}^t f_\omega)) \subset \mathbb{P}(\bigwedge^2 V),$$
where $\ker({}^t f_\omega)$ is the map defined in (2.4); therefore $\mathrm{codim}\Lambda_\omega = \mathrm{rank}\,\omega$.

(2) If $x \in \mathrm{Im}\,{}^t f_\omega \subset V^*$, then $\mathrm{codim}\Lambda_\omega^x = \mathrm{rank}\,\omega - 1$, i.e. Λ_ω is a hyperplane in Λ_ω^x.

(3) If $x \notin \mathrm{Im}\,{}^t f_\omega \subset V^*$, then $\mathrm{codim}\Lambda_\omega^x = \mathrm{rank}\,\omega$, i.e. $\Lambda_\omega = \Lambda_\omega^x$.

(4) If $\langle x, y\rangle \subset \mathrm{Im}\,{}^t f_\omega$, then $\mathrm{codim}\Lambda_\omega^{xy} = \mathrm{rank}\,\omega - 2$, i.e. Λ_ω^x is a hyperplane in Λ_ω^{xy}.

(5) $\mathrm{codim}\Lambda_{\omega_x} = n + \mathrm{rank}\,\omega_x$.

(6) If $\langle x, y\rangle \subset \mathrm{Im}\,{}^t f_\omega$, then $\mathrm{codim}\Lambda_{\omega,x\wedge y} = \mathrm{rank}\,\omega - 1$, i.e. $\Lambda_{\omega,x\wedge y}$ is a hyperplane in Λ_ω^{xy}.

For a general 3-form ω we immediately get:

Lemma 2.13. *If ω has rank $n + 1$, i.e. it satisfies (GC2), then $\{\Lambda_\omega^x \mid x \in V^*\}$ and $\{\Lambda_\omega^{xy} \mid x \wedge y \in \bigwedge^2 V^*\}$ are the sets of codimension n subspaces and codimension $n - 1$ subspaces of $\mathbb{P}(\bigwedge^2 V)$ that contain Λ_ω, respectively.*

Consider now the quadric $Q_{\omega\wedge x} \subset \mathbb{P}(\bigwedge^2 V)$, of equation $(\omega \wedge x)(L \wedge L) = 0$. Notice, first, that it depends only on the restriction ω_x of ω to V_x. In fact if $\omega = \omega_x + \beta_x \wedge x$, then $\omega \wedge x = \omega_x \wedge x$.

We first find the singular locus of $Q_{\omega\wedge x}$. Let $K_{\omega_x} = \ker f_{\omega_x} \subset V_x$, and set $\Lambda_{\omega_x}^K = \mathbb{P}(K_{\omega_x} \bigwedge V) \subset \mathbb{P}(\bigwedge^2 V)$.

Lemma 2.14. *The singular locus of $Q_{\omega\wedge x}$ is the subspace $\langle \Lambda_{\omega_x} \cup \Lambda_{\omega_x}^K\rangle \subset \mathbb{P}(\bigwedge^2 V)$. In particular, if ω_x has rank m, then $Q_{\omega\wedge x}$ has rank $2m$.*

Proof. The singular locus of $Q_{\omega\wedge x}$ is $\mathbb{P}(\ker\rho_{\omega\wedge x})$, where

$$\begin{aligned}\rho_{\omega\wedge x}\colon \bigwedge^2 V &\to \bigwedge^2 V^*\\ L &\mapsto (\omega\wedge x)(L),\end{aligned}$$

so as in the proof of Lemma 2.7, (3), we may assume $\omega = \omega_x \in \bigwedge^3 V_x^*$ and let $L = L_x + v_x \wedge e$, where $L_x \in \bigwedge^2 V_x$, $v_x \in V_x$ and $x(e) = 1$. Then the singular locus of $Q_{\omega\wedge x}$ is spanned by classes of 2-vectors L such that

$$(\omega\wedge x)(L) = \omega_x(L_x)\wedge x - \omega_x(v_x) = 0,$$

which implies

$$\omega_x(L_x) = \omega_x(v_x) = 0,$$

or equivalently

$$[L_x] \in \Lambda_{\omega_x} \qquad \text{and} \qquad [v_x\wedge e] \in \Lambda^K_{\omega_x}.$$

The rank of $Q_{\omega\wedge x}$ equals the rank of $\rho_{\omega\wedge x}$, which is $2\operatorname{rank}\omega_x$, by Lemma 2.73. □

Remark 2.15. When $f_{\omega_x} : V_x \to \bigwedge^2 V_x^*$ is injective, i.e. ω_x has rank n, then the singular locus of $Q_{\omega\wedge x}$ is Λ_{ω_x} and the rank of $Q_{\omega\wedge x}$ is $2n$.

We shall study now the linear spaces in the quadrics $Q_{\omega\wedge x}$, with special attention to those containing also Λ_ω.

Lemma 2.16. *The quadric $Q_{\omega\wedge x}$, for $x\in V^*$, contains the codimension n linear subspaces Λ^x_ω, $\Lambda_{\omega,x\wedge y}$ for $x\wedge y \in \bigwedge^2 V^*$, and $\mathbb{P}(\bigwedge^2 V_x)$.*

In particular, each quadric $Q_{\omega\wedge x}$, $x\in V^*$, contains Λ_ω, and each quadric of the pencil generated by $Q_{\omega\wedge x}$ and $Q_{\omega\wedge y}$ contains the linear subspace $\Lambda_{\omega,x\wedge y}$.

Proof. $[L]\in Q_{\omega\wedge x}$ if and only if

$$(\omega\wedge x)(L\wedge L) = (\omega(L)\wedge x - \omega(x(L))(L) = -2\omega(L)(x(L)) = 0,$$

which is equivalent to

$$(\omega(L)\wedge x)(L) = 0.$$

So if $\omega(L)\wedge x = 0$ or $x(L) = 0$, then $[L]\in Q_{\omega\wedge x}$. Therefore $\Lambda_{\omega\wedge x}\subset Q_{\omega\wedge x}$ and $\mathbb{P}(\bigwedge^2 V_x)\subset Q_{\omega\wedge x}$.

Similarly, if $x, y\in V^*$ and $\omega(L)\wedge x\wedge y = 0$, then $\omega(L) = ax+by$ for some $ax+by\in\langle x,y\rangle$. If furthermore $(x\wedge y)(L) = 0$, then

$$(\omega(L)\wedge(cx+dy))(L) = ((ax+by)\wedge(cx+dy))(L) = 0$$

so $[L]\in Q_{\omega\wedge(cx+dy)}$ for any $cx+dy\in\langle x,y\rangle$. Therefore the linear space $\Lambda_{\omega,x\wedge y}$ is contained in the pencil of quadrics generated by $Q_{\omega\wedge x}$ and $Q_{\omega\wedge y}$. □

Since the quadric $Q_{\omega\wedge x} = Q_{\omega'\wedge x}$, whenever $(\omega'-\omega)\wedge x = 0$, the Lemma 2.16 applies to show that

$$\Lambda^x_{\omega'}\subset Q_{\omega\wedge x}$$

for every $[\omega']$ in the set

$$\{[\omega']\in\mathbb{P}(\bigwedge^3 V^*) \mid \omega(L)\wedge\omega'(L) = 0 \quad\text{when}\quad x(L) = 0\}. \tag{2.11}$$

Likewise

$$\Lambda_{\omega',x\wedge y}\subset Q_{\omega\wedge x}$$

for every $[\omega']$ and $[y]\neq[x]$ such that

$$\{([\omega'],[y])\in\mathbb{P}(\bigwedge^3 V^*)\times\mathbb{P}(V_x^*) \mid \omega(L)\wedge\omega'(L) = 0 \quad\text{when}\quad (x\wedge y)(L) = 0\}. \tag{2.12}$$

When the restriction $\omega_x \in \bigwedge^3 V_x^*$ has rank n, then the quadric $Q_{\omega\wedge x}$ has rank $2n$, by Lemma 2.7 (3). Hence $Q_{\omega\wedge x}$ contains two $\binom{n}{2}$-dimensional families of linear spaces of minimal codimension n.

Theorem 2.17. *Let $\omega \in \bigwedge^3 V^*$ and $x \in V^*$ and assume that the restriction ω_x to V_x has rank n. Then:*

(1) The singular locus of the quadric $Q_{\omega\wedge x}$ is the linear space

$$\Lambda_{\omega_x} = \{[L] \in \mathbb{P}(\bigwedge^2 V) \mid x(L) = \omega_x(L) = 0\}.$$

(2) The two $\binom{n}{2}$-dimensional spinor varieties of n-codimensional linear subspaces in the quadric $Q_{\omega\wedge x} \subset \mathbb{P}(\bigwedge^2 V)$ are birationally parametrised by the sets (2.11) *and* (2.12) *respectively.*

(3) The linear space

$$\{[L] \in \mathbb{P}(\bigwedge^2 V) \mid x(L) = 0\} = \langle G(2, V_x)\rangle,$$

spanned by the subgrassmannian $G(2, V_x)$, is contained in $Q_{\omega\wedge x}$ and belongs to the spinor variety parametrised by (2.11) *if n is odd, and to the spinor variety parametrised by* (2.12) *if n is even.*

Proof. It remains, first, to show that the two families of linear subspaces (2.11) and (2.12) are $\binom{n}{2}$-dimensional. The linear spaces in $\Lambda^x_{\omega'}$ in (2.11) depend on a 2-form $\beta \in \bigwedge^2 V^*$ such that $\omega' = \omega_x + \beta \wedge e$, where $x(e) = 1$, so this family is $\binom{n}{2}$-dimensional.

The linear spaces $\Lambda_{\omega',x\wedge y}$ in (2.12) depend on the choice of y and the choice of $\beta' \wedge y$, such that $\omega' \in \langle \omega, \beta' \wedge y\rangle$. The first choice is $(n-1)$-dimensional, the second is $\binom{n-1}{2}$-dimensional, so they sum to $\binom{n}{2}$. The second statement of the theorem follows.

For the third statement, assume $x(L) = 0$. Then clearly

$$\omega(L)(x(L)) = 0$$

and so the linear space

$$\{[L] \mid x(L) = 0\}$$

of codimension n is also contained in $Q_{\omega\wedge x}$. The intersection

$$\{[L] \mid x(L) = 0\} \cap \{[L] \mid \omega(L) \wedge x = 0\} = \{[L] \in \mathbb{P}(\bigwedge^2 V_x) \mid \omega_x(L) = 0\}$$

has codimension n in $\{[L] \mid x(L) = 0\}$, so this last linear subspace belongs to the spinor variety parametrised by (2.11) if n is odd, and to the spinor variety parametrised by (2.12) if n is even. □

Remark 2.18. We conclude this section observing that two maximal isotropic spaces $\Lambda^x_{\omega'}$ and $\Lambda_{\omega',x\wedge y}$ of opposite families on $Q_{\omega\wedge x}$, obtained from the *same* 3-form ω', are both contained in the subspace

$$\Lambda^{xy}_{\omega'} = \{[L] \in \mathbb{P}(\bigwedge^2 V) \mid \omega'(L) \wedge x \wedge y = 0\}$$

of codimension $n-1$. So they intersect along a subspace having codimension 1 in both of them. This will come in hand in Section 7.

3. The congruence

In this section we introduce the congruence X_ω defined by a 3-form ω. We keep our notation V for a fixed vector space of dimension $n+1$ and $\mathbb{G} = G(2, V)$.

3.1. **The congruence as a linear section of the Grassmannian.** Now, our main object of study is the following subset of the Grassmannian $\mathbb{G}$:

$$X_\omega := \{[L] \in \mathbb{G} \mid \omega(L) = 0\} \subset \mathbb{P}(\bigwedge^2 V).$$

3.1.1. *The equations of the congruence.* We start by studying X_ω in coordinates. Our 3-form ω reads:

$$\omega = \sum_{0\le i<j<k\le n} a_{i,j,k} x_i \wedge x_j \wedge x_k \in \bigwedge^3 V^*. \tag{3.1}$$

If we write L as

$$L = \sum_{0\le a<b\le n} p_{a,b} e_a \wedge e_b \tag{3.2}$$

where the $p_{a,b}$'s in (3.2) satisfy the Plücker relations, then we require that

$$\omega(\sum_{0\le a<b\le n} p_{a,b} e_a \wedge e_b) = 0. \tag{3.3}$$

More explicitly, we get

$$\omega(L) = \sum_{i,j,k} ((-1)^{i+j-1}(a_{i,j,k} - a_{i,k,j} + a_{k,i,j}))x_k p_{i,j} = 0. \tag{3.4}$$

Therefore, we deduce that the equations of X_ω are

$$\sum_{0\le i<j\le n} ((-1)^{i+j-1}(a_{i,j,k} - a_{i,k,j} + a_{k,i,j}))p_{i,j} = 0, \qquad k = 0, \dots, n, \tag{3.5}$$

i.e. we have $n+1$ linear equations, together with the Plücker relations. The equations are sometimes more convenient in the form

$$\sum_{0\le i<j\le n} ((-1)^{i+j+k-1}(a_{i,j,k} - a_{i,k,j} + a_{k,i,j}))p_{i,j} = 0, \qquad k = 0, \dots, n, \tag{3.6}$$

Remark 3.1. Note that the equations (3.5) define the linear span of X_ω, $\langle X_\omega \rangle$, which was denoted by Λ_ω in (2.6).

The linear space generated by equations (3.5) has a natural embedding in $\mathbb{P}(\bigwedge^2 V)^*$ as the linear subspace P_ω, orthogonal to the linear span of X_ω in the Plücker embedding, i.e.

$$P_\omega := \Lambda_\omega^\perp \tag{3.7}$$

(see section 2.2).

By Proposition 2.11, if $n \ge 4$ and the genericity condition (GC2) holds, these equations are linearly independent, therefore $\dim P_\omega = n$, while if $n = 3$, $\dim P_\omega = 2$. Let us see the embedding of P_ω in coordinates: to obtain the parametric equations of P_ω we simply consider the coefficients of the $p_{i,j}$'s in $\omega(L) = 0$, i.e. in equation (3.4):

$$\sum_k ((-1)^{i+j-1}(a_{i,j,k} - a_{i,k,j} + a_{k,i,j}))x_k;$$

therefore, if $q_{i,j}$ are the dual coordinates on $\mathbb{P}(\bigwedge^2 V)^*$, the parametric equations of P_ω are given by

$$q_{i,j} = \sum_k ((-1)^{i+j-1}(a_{i,j,k} - a_{i,k,j} + a_{k,i,j}))x_k. \tag{3.8}$$

In other words, if ω has rank $n+1$, i.e. the linear map $f_\omega : V \mapsto \bigwedge^2 V^*; v \mapsto \omega(v)$ is injective (GC2), we have a linear embedding

$$\mathbb{P}(f_\omega) \colon \mathbb{P}(V) \to \mathbb{P}(\bigwedge^2 V)^* \tag{3.9}$$

defined by (3.8). This map is represented by the following $(n+1)\times(n+1)$ skew-symmetric matrix of linear forms on $\mathbb{P}(V)$:

$$M_\omega := \left(\sum_k((-1)^{i+j-1}(a_{i,j,k} - a_{i,k,j} + a_{k,i,j}))x_k\right)_{\substack{i=0,\dots,n\\ j=0,\dots,n}} \tag{3.10}$$

using the usual convention, in (3.8), that $q_{i,j} = -q_{j,i}$.

Summarising, if ω has rank $n+1$, then

$$\Lambda_\omega^\perp = P_\omega = \mathrm{Im}(\mathbb{P}(f_\omega)) \cong \mathbb{P}(V). \tag{3.11}$$

Clearly, $P_\omega = \overline{\mathrm{Im}(\mathbb{P}(f_\omega))}$ always holds true (i.e. also if f_ω is not injective).

3.1.2. *The tangent space of the congruence.* To interpret X_ω geometrically, we consider certain linear subspaces of $\mathbb{P}(\bigwedge^3 V^*)$.

Fix a decomposable tensor $L \in \bigwedge^2 V$ and let $L^* \in \bigwedge^{n-1} V^*$ be its dual. We shall denote by $\mathbb{T}_L \subset \mathbb{P}(\bigwedge^3 V^*)$ the linear span of the union of all the embedded tangent spaces to $G(3, V^*)$ at the points corresponding to the 3-spaces π^* such that L^* contains π^*, i.e. $\mathbb{T}_L = \langle \mathbb{T}_{\pi^*} G(3, V^*) \mid \pi^* \subset L^*\rangle$.

Lemma 3.2. *Assume $n \geq 4$, and let $\omega \in \bigwedge^3 V^*$. Let $L = e \wedge f \in \bigwedge^2 V$. We fix a basis of V, $e_0, \dots, e_n$, with $e_0 = e$, $e_1 = f$, and the dual basis $x_0, \dots, x_n$. Then the following are equivalent:*

1. *$[L] \in X_\omega$;*
2. *ω can be uniquely written as $\omega = \omega_{01} + \beta_0 \wedge x_0 + \beta_1 \wedge x_1$, with β_0, $\beta_1 \in \bigwedge^2\langle x_2, \dots, x_n\rangle$, $\omega_{01} \in \bigwedge^3\langle x_2, \dots, x_n\rangle$;*
3. *$[\omega] \in \mathbb{T}_L$.*

Proof. $[L] \in \mathbb{G}$ has coordinates $p_{0,1} \neq 0$ and $p_{i,j} = 0$ if $(i, j) \neq (0, 1)$. Therefore, from (3.5), we deduce that $a_{0,1,h} = 0$, $\forall h = 0, \dots, n$, and (3.1) becomes:

$$\omega = \sum_{\substack{0\leq i<j<k\leq n\\ (i,j)\neq(0,1)}} a_{i,j,k} x_i \wedge x_j \wedge x_k = \beta_0 \wedge x_0 + \beta_1 \wedge x_1 + \omega_{01},$$

where we have set

$$\beta_0 := -\sum_{2\leq j<k\leq n} a_{0,j,k} x_j \wedge x_k, \qquad \beta_1 := -\sum_{2\leq j<k\leq n} a_{1,j,k} x_j \wedge x_k,$$

$$\omega_{01} := \sum_{2\leq i<j<k\leq n} a_{i,j,k} x_i \wedge x_j \wedge x_k.$$

Therefore the equivalence of (1) and (2) is proved. For the last equivalence, observe that $L^* = x_2 \wedge \dots \wedge x_n$, so, up to a coordinate change, a 3−space $\pi^* \subset L^*$ can be expressed as

$$[\pi^*] = [x_{n-2} \wedge x_{n-1} \wedge x_n] \in G(3, V^*) \subset \mathbb{P}(\bigwedge^3 V^*),$$

and the embedded tangent space to $G(3, V^*)$ at π^* can be expressed as

$$\begin{aligned}\mathbb{T}_{\pi^*} G(3, V^*) = \langle &[(\sum_{i=0}^{n-2} a_i x_i) \wedge x_{n-1} \wedge x_n],\\ &[(\sum_{i=0}^{n-3} b_i x_i - b_{n-1}x_{n-1}) \wedge x_{n-2} \wedge x_n], [(\sum_{i=0}^{n-3} c_i x_i + c_n x_n) \wedge x_{n-2} \wedge x_{n-1}]\rangle\end{aligned}$$

from which the last equivalence follows. □

Corollary 3.3. *Let $L \subset V$ be a 2-vector subspace, with $n \geq 4$; then*

$$\dim(\mathbb{T}_L) = \frac{n+3}{3}\binom{n-1}{2} - 1. \tag{3.12}$$

Proof. It follows from the equivalence of (3) and (2) of Lemma 3.2:

$$\dim(\mathbb{T}_L) = 2\binom{n-1}{2} + \binom{n-1}{3} - 1 = \frac{n+3}{3}\binom{n-1}{2} - 1.$$

□

Recall from §1.1, that a *linear congruence* is a congruence obtained by proper intersection of $\mathbb{G}$ with a linear subspace of $\mathbb{P}(\bigwedge^2 V)$ of codimension $n-1$, and that the *order* of a congruence Γ is the number of lines of Γ passing through a general point of $\mathbb{P}(V)$.

Theorem 3.4. *Let $\omega \in \bigwedge^3 V^*$, with $n = \dim \mathbb{P}(V) \geq 3$, be a general 3-form. Then $X_\omega \subset \mathbb{G}$ has dimension $n-1$, i.e. it is a congruence. Moreover, X_ω is contained in a reducible linear congruence.*

Proof. We give here an elementary proof of the Theorem which works for $n \geq 6$. For $n \leq 5$, see Examples 4.9, 4.10, 4.11 in Section 4.4. A different proof of the Theorem will be given in Section 3.2.

By case (3) of Lemma 3.2 we need to find how many spaces of the form $\mathbb{T}_L$ pass through ω. First of all, we observe that two (general) spaces $\mathbb{T}_L$ and $\mathbb{T}_{L'}$ are in general position; in fact, if $L = \langle e_0, e_1 \rangle$ and $L' = \langle e_2, e_3 \rangle$, then, by the equivalence of cases (2) and (3) of Lemma 3.2, a 3-form $[\omega] \in \mathbb{P}(\bigwedge^3 V^*)$ belongs to $\mathbb{T}_L \cap \mathbb{T}_{L'}$ if and only if we can write

$$\omega = x_0 \wedge x_2 \wedge \phi_{02} + x_0 \wedge x_3 \wedge \phi_{03} + x_1 \wedge x_2 \wedge \phi_{12} + x_1 \wedge x_3 \wedge \phi_{13} + \\ + x_0 \wedge \beta_0 + x_1 \wedge \beta_1 + x_2 \wedge \beta_2 + x_3 \wedge \beta_3 + \omega',$$

where, as in Lemma 3.2, $\beta_0, \dots, \beta_3 \in \bigwedge^2 \langle x_4, \dots, x_n \rangle$ and $\omega' \in \bigwedge^3 \langle x_4, \dots, x_n \rangle$.

From this, we infer that

$$\dim(\mathbb{T}_L \cap \mathbb{T}_{L'}) + 1 = 4(n-3) + 4\binom{n-3}{2} + \binom{n-3}{3} = \\ = \frac{(n-3)(n-1)(n+4)}{6},$$

from which we deduce

$$\begin{aligned} \dim(\langle \mathbb{T}_L, \mathbb{T}_{L'} \rangle) + 1 &= \dim(\mathbb{T}_L) + \dim(\mathbb{T}_{L'}) - \dim(\mathbb{T}_L \cap \mathbb{T}_{L'}) + 1 \\ &= 2\frac{n+3}{3}\binom{n-1}{2} - \frac{(n-3)(n-1)(n+4)}{6} \\ &= \binom{n+1}{3}, \end{aligned}$$

i.e. $\langle \mathbb{T}_L, \mathbb{T}_{L'} \rangle = \mathbb{P}(\bigwedge^3 V^*)$, and $\mathbb{T}_L$ and $\mathbb{T}_{L'}$ are in general position.

Then, we observe that there is a family of dimension $\dim \mathbb{G} = 2(n-1)$ of $\mathbb{T}_L$'s, therefore, recalling (3.12), if ω is general, the dimension of the $\mathbb{T}_L$'s passing through ω is equal to

$$\dim \mathbb{G} + \dim(\mathbb{T}_L) - \dim(\mathbb{P}(\bigwedge^3 V^*) = 2(n-1) + \frac{n+3}{3}\binom{n-1}{2} - \binom{n+1}{3} \\ = n-1.$$

By (3.5), it follows that X_ω is contained in a linear congruence. □

3.1.3. *More genericity conditions.* Theorem 3.4 naturally gives rise to the following genericity conditions, which we call 4 and 5, after §2.1.

Definition 3.5. We shall consider the following conditions on a 3-form $\omega \in \bigwedge^3 V^*$:

(GC4): X_ω has the expected dimension $n-1$;
(GC5): X_ω satisfies (GC4) and it is smooth.

Remark 3.6. The condition (GC4) implies the condition (GC2) in 2.1.

(1) When ω has rank m, then the linear span of X_ω has codimension m in $\mathbb{P}(\bigwedge^2 V)$ and the codimension of X_ω in $\mathbb{G}$ is at most $m-2$.
(2) For $n \geq 8$, the 3-form $\omega = x_0 \wedge x_1 \wedge x_2 + x_3 \wedge x_4 \wedge x_5 + x_6 \wedge x_7 \wedge x_8$, has rank 9, while X_ω has codimension 6, so (GC2) holds, while (GC4) fails for ω.

3.1.4. *The order of the congruence.* We compute now the order of the congruence X_ω.

Proposition 3.7. *If ω is general, $X_\omega \subset \mathbb{G}$ is a congruence of order zero if n is even and of order one if n is odd.*

Proof. Let $n \geq 6$. We look at the intersection $\Sigma_P \cap X_\omega$, where, without loss of generality, we can suppose that $P = [1, 0, \cdots, 0] = [e_0]$. Σ_P is given by the lines of type $[L] = [e_0 \wedge f]$, with e_0 and f linearly independent; so Σ_P is defined, in the Plücker coordinates, by $p_{i,j} = 0$ if $i > 0$.

Therefore, by (3.6), the intersection $\Sigma_P \cap X_\omega$ is defined by the n equations:

$$\sum_{0<j\leq n} ((-1)^{j+k-1}(a_{0,j,k} - a_{0,k,j}))p_{0,j} = 0 \tag{3.13}$$

where $k = 1, \ldots, n$, i.e. we have n linear equations in the n indeterminates $p_{0,1}, \ldots, p_{0,n}$. Let A be the matrix associated to the homogeneous linear system (3.13); then A is skew-symmetric, and therefore, if ω is general, the system (3.13) has only the zero solution if n is even and has a vector space of solutions of dimension one if n is odd.

If $n \leq 5$, see the examples in Section 4.4.

□

3.2. **The congruence X_ω as a degeneracy locus.** We recall now some facts about vector bundles on Grassmannians, and we apply them to the study of the congruence X_ω. We recall the universal exact sequence (1.1) on the Grassmannian $\mathbb{G} = G(2, V)$:

$$0 \to \mathcal{U} \to V \otimes \mathcal{O}_{\mathbb{G}} \to \mathcal{Q} \to 0.$$

Recall that, for the Plücker embedding, $\mathcal{O}_{\mathbb{G}}(1) \cong \det(\mathcal{U}^*) \cong \det(\mathcal{Q})$. Over $\mathbb{G}$, we have also $H^0(\mathcal{Q}) = V$, $H^0(\mathcal{U}^*) = V^*$. We refer to [Wey03] for terminology and basic results on homogeneous bundles.

Theorem 3.8. *The congruence X_ω is the zero locus of a section of $\mathcal{Q}^*(1)$. If ω is general enough, X_ω is smooth of the expected dimension $n-1$, i.e. X_ω satisfies (GC5).*

Proof. We use the theorem of Borel-Bott-Weil, in particular [Wey03, Theorem 4.1.8].

Here, this gives a natural isomorphism

$$H^0(\mathcal{Q}^*(1)) \simeq \bigwedge^3 V^*.$$

Therefore a 3-form ω on V can be interpreted as a global section of $\mathcal{Q}^*(1)$ and it defines a bundle map

$$\mathcal{O}_{\mathbb{G}} \xrightarrow{\varphi_\omega} \mathcal{Q}^*(1). \tag{3.14}$$

The degeneracy locus of φ_ω is therefore X_ω, which was defined in Section 3.1 as $\{[L] \in \mathbb{G} \mid \omega(L) = 0\}$. By a Bertini type theorem, for general ω, X_ω has codimension equal to $\operatorname{rank} \mathcal{Q}^*(1) = n-1$, and it is smooth. Moreover also $\dim X_\omega = n-1$. □

From Theorem 3.8 we deduce further global properties of X_ω.

Theorem 3.9. *If ω is an alternating 3-form on $\mathbb{P}(V)$ that satisfies (GC4), then X_ω is a Fano variety of index 3 and dimension $n-1$ with Gorenstein singularities, which is smooth if ω is general (GC5). Moreover the sheaf $\mathcal{O}_{X_\omega}$ has the Koszul locally free resolution:*

$$0 \to \mathcal{O}_{\mathbb{G}}(2-n) \to \bigwedge^{n-2}(\mathcal{Q}(-1)) \to \cdots \to \mathcal{Q}(-1) \xrightarrow{{}^t\varphi_\omega} \mathcal{O}_{\mathbb{G}} \to \mathcal{O}_{X_\omega} \to 0. \tag{3.15}$$

Proof. We note that the Koszul complex associated with the section φ_ω of $\mathcal{Q}^*(1)$ defining X_ω is exact. The resolution follows observing that $\det(\mathcal{Q}^*(1))^* \cong \det(\mathcal{Q}(-1)) \cong \mathcal{O}_{\mathbb{G}}(-n+2)$. Then by adjunction, the canonical sheaf ω_{X_ω} of X_ω is locally free of rank 1 and has the following expression

$$\omega_{X_\omega} \cong \omega_{\mathbb{G}} \otimes \bigwedge^{n-1}(\mathcal{Q}^*(1))|_{X_\omega} \cong \mathcal{O}_{\mathbb{G}}(-n-1+n-2)|_{X_\omega} = \mathcal{O}_{X_\omega}(-3).$$

□

Corollary 3.10. *In the hypothesis of the preceding theorem, X_ω is arithmetically Cohen-Macaulay (aCM for short, in what follows) in its linear span Λ_ω and arithmetically Gorenstein (aG for short).*

Proof. The first statement follows from resolution (3.15), because the exterior powers of $\mathcal{Q}$ are aCM by Bott's theorem (cf. [Wey03, Ch. 4]). Since X_ω is also subcanonical by Theorem 3.9, then it is aG by [Mig98, Proposition 4.1.1]. □

3.2.1. *The cohomology class of the congruence.* The cohomology class of X_ω is given by Porteous formula:

$$[X_\omega] = c_{n-1}(\mathcal{Q}^*(1)) \cap [\mathbb{G}].$$

If

$$P^0 \subset P^1 \subset \cdots \subset P^{n-1-i} \subset \cdots \subset P^{n-j} \subset \cdots \subset P^n = \mathbb{P}(V)$$

is a complete flag, the cohomology ring of $\mathbb{G}$ has basis $\sigma_{i,j}$, $i = 0, \dots, n-1$, $j = 0, \dots, i$, where

$$\sigma_{i,j} \cap [\mathbb{G}] = [\{[L] \in \mathbb{G} | L \subset P^{n-j}, L \cap P^{n-1-i} \neq \emptyset\}].$$

Then

$$c_i(\mathcal{Q}) = \sigma_i := \sigma_{i,0},$$

and therefore $c_i(\mathcal{Q}^*) = (-1)^i \sigma_i$. As above, we write formally the Chern polynomial of $\mathcal{Q}^*$

$$c_t(\mathcal{Q}^*) = \prod_{i=1}^{n-1}(1 - a_i t),$$

where $a_1, \dots, a_{n-1}$ are formal symbols. Since $\sigma_1 \cap [\mathbb{G}]$ is the class of a hyperplane section of the Plücker embedding, we have also $c_1(\mathcal{O}_{\mathbb{G}}(1)) = \sigma_1$, and we get

$$c_t(\mathcal{Q}^*(1)) = (1 + (a_1 + \sigma_1)t) \cdots (1 + (a_{n-1} + \sigma_1)t),$$

and therefore, applying [DeP03, Lemma 2.1],

$$\begin{aligned} c_{n-1}(\mathcal{Q}^*(1)) = (a_1 + \sigma_1) \cdots (a_{n-1} + \sigma_1) &= \sum_{\ell=0}^{n-1} (-1)^{n-1-\ell} \sigma_{n-1-\ell} \sigma_1^\ell \\ &= \sum_{\ell=0}^{n-1} (-1)^{n-1-\ell} \sigma_{n-1-\ell} \Big(\sum_{i=0}^{[\frac{\ell}{2}]} \left(\binom{\ell}{i} \cdot \frac{\ell - 2i + 1}{\ell - i + 1} \right) \sigma_{\ell-i,i} \Big) \\ &=: \sum_{\ell=0}^{[\frac{n-1}{2}]} d_\ell(n) \sigma_{n-1-\ell,\ell}. \end{aligned}$$

Here the integers $d_\ell(n)$ are defined by the last equality, and their collection is called the *multidegree of* X_ω. By Poincaré duality

$$\int_{[\mathbb{G}]} \sigma_{n-1-\ell,\ell}\sigma_{n-1-\ell',\ell'} = \delta_{\ell,\ell'}$$

so the multidegree of X_ω is also defined by

$$d_\ell(n) = \int_{[X_\omega]} \sigma_{n-1-\ell,\ell} \qquad\qquad 0 \le \ell \le n-1.$$

Lemma 3.11. *The multidegree* $(d_\ell(n))$*,* $\ell = 0,\dots,n-1$ *satisfies the initial condition*

$$d_0(2m) = 0, \qquad d_0(2m-1) = 1, \qquad m = 2,3,4,\dots$$

and the recursion relation

$$d_\ell(n) = d_{\ell-1}(n-1) + d_\ell(n-1)$$

when $\ell = 1,2,\dots,\left[\frac{n-1}{2}\right]$.

Proof. The degree $d_0(n)$ is simply the order of the congruence X_ω, so the initial condition follows from Proposition 3.7.

Next, recall, from Schubert calculus, that $\sigma_{i,j}\sigma_1 = \sigma_{i+1,j} + \sigma_{i,j+1}$ (when $i > j$). Let $\omega = \omega_x + \beta_x \wedge x$. We choose a flag such that $P^{n-\ell} \subset \mathbb{P}(V_x) \subset \mathbb{P}(V)$. If $\ell > 0$, then $d_\ell(n) = \deg X_\omega \cap Z_\ell(n)$ where

$$Z_\ell(n) = \{[L] | L \subset P^{n-\ell}, L \cap P^\ell \neq \emptyset\} \subset G(2,V),$$

and has class

$$[Z_\ell(n)] = \sigma_{n-1-\ell,\ell} \cap [G(2,V)].$$

But $P^{n-\ell} \subset \mathbb{P}(V_x)$, so $Z_\ell(n) \subset G(2,V_x)$. The class of this subvariety in $G(2,V_x)$ is

$$[Z_\ell(n)] = \sigma_{n-2-\ell,\ell-1} \cap [G(2,V_x)],$$

while

$$[X_\omega] \cap [G(2,V_x)] = [\{[L] | \omega_x(L) = \beta_x(L) = 0\}] = [X_{\omega_x}] \cap [H_{\beta_x}] \cap [G(2,V_x)],$$

where clearly H_{β_x} is the hyperplane defined by $\{\beta_x(L) = 0\}$. Computing the degree of $X_\omega \cap Z_\ell(n)$ on $G(2,V)$ and on $G(2,V_x)$ we get

$$\begin{aligned} d_\ell(n) &= \int_{[X_{\omega_x}]} \sigma_{n-2-\ell,\ell-1}\sigma_1 = \int_{[X_{\omega_x}]} (\sigma_{n-1-\ell,\ell-1} + \sigma_{n-2-\ell,\ell}) \\ &= d_{\ell-1}(n-1) + d_\ell(n-1). \end{aligned}$$

□

For low values of n, $3 \le n \le 9$, we get the following multidegree for X_ω:

(3.16) $\quad (1,0), \quad (0,1), \quad (1,1,1), \quad (0,2,2), \quad (1,2,4,2), \quad (0,3,6,6), \quad (1,3,9,12,6).$

The recursion of Lemma 3.11, and the initial degrees of (3.16), may be displayed in a triangle with initial entries

$$a_{(2k+1,0)} = 1, \qquad a_{(2k,0)} = 0, \qquad k = 0,1,2,\dots$$

and

$$a_{(i,j)} = a_{(i,j-1)} + a_{(i-1,j)} \qquad i = 1,2,\dots, \text{ and } j = 1,2,\dots,i.$$

The multidegree of X_ω is identified as $(d_\ell(n)) = (a_{(n-1-\ell,\ell)}), \ell = 0,...,n-1$.

$$
(a_{i,j}) = \begin{array}{cccccccccc}
1 \\
0 & 0 \\
1 & 1 & 1 \\
0 & 1 & 2 & 2 \\
1 & 2 & 4 & 6 & 6 \\
0 & 2 & 6 & 12 & 18 & 18 \\
1 & 3 & 9 & 21 & 39 & 57 & 57 \\
0 & 3 & 12 & 33 & 72 & 129 & 186 & 186 \\
1 & 4 & 16 & 49 & 121 & 250 & 436 & 622 & 622 \\
0 & 4 & 20 & 69 & 190 & 440 & 876 & 1498 & 2120 & 2120
\end{array}. \tag{3.17}
$$

In fact

$$a_{(n-1-\ell,\ell)} = d_\ell(n) = d_{\ell-1}(n-1) + d_\ell(n-1) = a_{(n-1-\ell,\ell-1)} + a_{(n-2-\ell,\ell)}.$$

3.2.2. *The degree of the congruence and the Fine numbers.* The Fine numbers form the sequence $1, 0, 1, 2, 6, 18, 57, 186, \dots$ see [Fine, DS01].

Proposition 3.12. *The degree of $X_\omega \subset \mathbb{G}$ is the n-th Fine number that can be read off as the $(n+1)$-st diagonal element*

$$\deg X_\omega = a_{(n-1,n-1)} = a_{(n-1,n-2)} = d_n(2n-1)$$

in the above triangle of numbers. The multidegree of X_ω is given by the antidiagonals

$$(d_0(2m), d_1(2m), \dots, d_{m-1}(2m)) = (a_{(2m-1,0)}, a_{(2m-2,1)}, \dots, a_{(m,m-1)})$$

when $n = 2m$ and

$$(d_0(2m-1), d_1(2m-1), \dots, d_{m-1}(2m-1)) = (a_{(2m-2,0)}, a_{(2m-3,1)}, \dots, a_{(m-1,m-1)}).$$

when $n = 2m-1$.

Proof. It remains to show that

$$\deg X_\omega = d_m(2m-1).$$

Let $\dim V = 2m$, and let $\omega \in \bigwedge^3 V^*$ be a general 3-form. Assume $V_0^* \subset V^*$ is a general subspace of dimension $m+1$, and choose $x_{m+1}, \dots, x_{2m-1} \subset V^*$ such that

$$V^* = V_0^* \oplus \langle x_{m+1}, \dots, x_{2m-1} \rangle.$$

Then

$$\omega = \omega_0 + x_{m+1} \wedge \beta_{m+1} + \dots + x_{2m-1} \wedge \beta_{2m-1} + \alpha$$

where $\omega_0 \in \bigwedge^3 V_0^*$, $\beta_i \in \bigwedge^2 V_0^*$, and $\alpha \in \bigwedge^2 \langle x_{m+1}, \dots, x_{2m-1} \rangle \wedge V^*$. Let $P^m = \{x_{m+1} = \dots = x_{2m-1} = 0\} \subset \mathbb{P}(V)$ and let $Z_{m-1}(2m-1) = \{[L] | L \subset P^m\}$. Then

$$\begin{aligned} X_\omega \cap Z_{m-1}(2m-1) &= \{[L] | L \subset P^m, \omega_0(L) = \beta_{m+1}(L) = \dots = \beta_{2m-1}(L) = 0\} \\ &= X_{\omega_0} \cap H_{\beta_{m+1}} \cap \dots \cap H_{\beta_{2m-1}}. \end{aligned}$$

So

$$\deg X_{\omega_0} = \deg(X_\omega \cap Z_{m-1}(2m-1)) = d_m(2m-1),$$

and the proposition follows. □

4. The fundamental locus

Definition 4.1. The *fundamental locus* of a congruence X is the set of points of $\mathbb{P}(V)$ belonging to infinitely many lines of X.

It is clear that if X has order zero, then its fundamental locus is simply the union of the lines of X. We will see in a minute that the fundamental locus of the congruence X_ω has a natural structure of scheme as a suitable degeneracy locus.

4.1. **A skew symmetric matrix associated with a trilinear form.** The 3-form $\omega \in \bigwedge^3 V^*$ defines a natural map of vector bundles on $\mathbb{P}(V)$. In this section we identify this map and its degeneracy loci on $\mathbb{P}(V)$.

The first observation is that there is a natural isomorphism:

$$H^0(\Omega^2_{\mathbb{P}(V)}(3)) \cong \bigwedge^3 V^*.$$

This is provided again by Borel-Bott-Weil's theorem. Now, in view of the natural isomorphisms

$$\begin{aligned} H^0(\Omega^2_{\mathbb{P}(V)}(3)) \subset & H^0(\Omega^1_{\mathbb{P}(V)} \otimes \Omega^1_{\mathbb{P}(V)}(3)) \cong \mathrm{Hom}(\Omega^1_{\mathbb{P}(V)}(1)^*, \Omega^1_{\mathbb{P}(V)}(2)) \\ & \cong \mathrm{Hom}(\mathcal{T}_{\mathbb{P}(V)}(-1), \Omega^1_{\mathbb{P}(V)}(2)), \end{aligned}$$

the 3-form $\omega \in \bigwedge^3 V^*$ determines a bundle map:

$$\phi_\omega \colon \mathcal{T}_{\mathbb{P}(V)}(-1) \to \Omega^1_{\mathbb{P}(V)}(2). \tag{4.1}$$

Note that ϕ_ω is skew-symmetric, in the sense that

$$\phi^*_\omega = -\phi_\omega(1),$$

where $\phi^*_\omega : \mathcal{T}^1_{\mathbb{P}(V)}(-2) \to \Omega_{\mathbb{P}(V)}(1)$ is the dual map of ϕ_ω. Indeed, by the above description, $H^0(\Omega^2_{\mathbb{P}(V)}(3))$ is just the skew-symmetric part of $\mathrm{Hom}(\mathcal{T}_{\mathbb{P}(V)}(-1), \Omega^1_{\mathbb{P}(V)}(2))$, and the map induced by ϕ_ω on the global sections

$$f_\omega \colon H^0(\mathcal{T}_{\mathbb{P}(V)}(-1)) \to H^0(\Omega^1_{\mathbb{P}(V)}(2))$$

is the map f_ω of (2.5).

We can interpret ϕ_ω in more concrete terms, in the following way: from Euler sequence twice we get the diagram

$$\begin{array}{ccc} \mathcal{T}_{\mathbb{P}(V)}(-1) & \xrightarrow{\phi_\omega} & \Omega^1_{\mathbb{P}(V)}(2) \\ \uparrow & & \downarrow \\ V \otimes \mathcal{O}_{\mathbb{P}(V)} & \xrightarrow{M_\omega} & V^* \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \end{array} \tag{4.2}$$

where M_ω is obtained by composition, so we can think of M_ω as a $(n+1) \times (n+1)$ skew-symmetric matrix with linear entries on $\mathbb{P}(V)$. In fact, this matrix M_ω in suitable coordinates is the matrix defined in (3.10).

Remark 4.2. Since ϕ_ω is a skew-symmetric map between two bundles of rank n, M_ω has rank at most n when n is even, and $n-1$ when n is odd. We will see in next Section that, if ω is general, then these are the generic ranks of M_ω.

The map ϕ_ω will be considered again in Section 4.5, where we will describe its kernel and cokernel both in the cases n even and n odd.

4.2. **Degeneracy loci.** Let us now study the degeneracy loci of ϕ_ω (or, which is the same, of M_ω). Let us denote by $M_r = \{P \in \mathbb{P}(V) \mid \operatorname{rank} \varphi_{\omega|P} \leq r\}$ the locus of the points where ϕ_ω has rank at most r.

We endow M_r with its natural scheme structure, given by the principal Pfaffians, $(r+2) \times (r+2)$ if r is even, and $(r+1) \times (r+1)$ if r is odd. Notice that in this last case $M_r = M_{r-1}$. Moreover, by Remark 4.2, $\mathbb{P}(V) = M_n$ if n is even and $\mathbb{P}(V) = M_{n-1}$ if n is odd.

The degeneracy loci of a (twisted) skew-symmetric map of vector bundles is studied in [HT84]: in particular, in [HT84, Theorem 10(b)] the cohomology class of each degeneracy locus is computed. In our setting, the class of M_r, if r is even, is

$$[M_r] = \det \begin{pmatrix} c_{n-r-1} & c_{n-r} & \cdots & \\ c_{n-r-3} & c_{n-r-2} & & \\ \vdots & & \ddots & \\ & & & c_1 \end{pmatrix} \tag{4.3}$$

where $c_i = c_i(\Omega^1_{\mathbb{P}(V)}(1) \otimes \sqrt{\mathcal{O}_{\mathbb{P}(V)}(1)})$ and $\sqrt{\mathcal{O}_{\mathbb{P}(V)}(1)}$ has to be thought, formally, by the *squaring principle*, as a line bundle such that $\sqrt{\mathcal{O}_{\mathbb{P}(V)}(1)} \otimes \sqrt{\mathcal{O}_{\mathbb{P}(V)}(1)} = \mathcal{O}_{\mathbb{P}(V)}(1)$.

Now, $c_t(\mathcal{O}_{\mathbb{P}(V)}(1)) = 1 + ht$, where c_t as usual denotes the Chern polynomial and h is the hyperplane class, so, if we put $c_t(\sqrt{\mathcal{O}_{\mathbb{P}(V)}(1)}) = 1 + at$, where a is a formal symbol, we have $1 + ht = (1 + 2at)$, which implies that $c_t(\sqrt{\mathcal{O}_{\mathbb{P}(V)}(1)}) = 1 + \frac{h}{2}t$; in other words, in expression (4.3), we can write $c_i = c_i(\Omega^1_{\mathbb{P}(V)} \otimes \mathcal{O}_{\mathbb{P}(V)}(\frac{3}{2}))$.

We recall, for example by the Euler sequence, that

$$c_t(\Omega^1_{\mathbb{P}(V)}) = (1 - ht)^{n+1}$$

and therefore

$$c_i(\Omega^1_{\mathbb{P}(V)}) = (-1)^i \binom{n+1}{i} h^i, \qquad i = 0, \dots, n.$$

On the other hand, it is easy to see, reasoning as above, that $c_t(\mathcal{O}_{\mathbb{P}(V)}(\frac{3}{2})) = 1 + \frac{3}{2}ht$.

Finally, if we write formally $c_t(\Omega^1_{\mathbb{P}(V)}) = \prod_{i=1}^n (1 + a_i ht)$, recalling the formula for the Chern polynomial of a tensor product, we obtain

$$\begin{aligned} c_t(\Omega^1_{\mathbb{P}(V)} \otimes \mathcal{O}_{\mathbb{P}(V)}(\frac{3}{2})) &= \prod_{i=1}^n (1 + (a_i + \frac{3}{2})ht \\ &= \left(\sum_{k=0}^{i} (-1)^k \frac{3^{i-k}(1+k)}{2^{i-k}(n+1-k)} \binom{i+1}{k+1} \right) \binom{n+1}{i+1} h^i. \end{aligned}$$

In particular

$$\begin{aligned} c_1 &= (\frac{n}{2} - 1)h, \\ c_2 &= \frac{n^2 - 5n + 12}{8} h^2, \\ c_3 &= \frac{n^3 - 9n^2 + 44n - 108}{48} h^3, \qquad \text{etc.} \end{aligned}$$

4.2.1. *Even n.* In this case

$$[M_{n-2}] = c_1(\Omega^1_{\mathbb{P}(V)} \otimes \mathcal{O}_{\mathbb{P}(V)}(\frac{3}{2})) = (\frac{n}{2} - 1)h$$

i. e. M_{n-2} is a hypersurface of degree $\frac{n}{2} - 1$.

By a Bertini type theorem, its singular locus is contained in M_{n-4} and it is equal to M_{n-4} if ω is general, for which we have

$$[M_{n-4}] = \det \begin{pmatrix} c_3 & c_4 & c_5 \\ c_1 & c_2 & c_3 \\ 0 & 1 & c_1 \end{pmatrix} = c_1(c_2c_3 - c_1c_4 + c_5) - c_3^2$$

and in particular it has codimension 6 in $\mathbb{P}(V)$; therefore, M_{n-2} is smooth up to $\mathbb{P}^4$, and we expect that it is singular of dimension 0 in $\mathbb{P}^6$ and of dimension 2 in $\mathbb{P}^8$. But, making explicit calculations, we obtain

$$[M_{n-4}] = \frac{(n)(n-6)(n+1)(n+2)(n^2 - 9n + 44)}{2880} h^6$$

and therefore M_{n-2} is smooth also in $\mathbb{P}^6$. Actually for $n = 6$ we get a smooth quadric 5-fold as degeneracy locus, as we shall see in Example 4.12.

Moreover M_{n-6} has codimension 15, so M_{n-4} is smooth up to $n = 14$.

4.2.2. *Odd n.* In this case we have

(4.4)
$$[M_{n-3}] = \det\begin{pmatrix} c_2 & c_3 \\ 1 & c_1 \end{pmatrix} = c_1c_2 - c_3 = \frac{n^3 - 6n^2 + 11n + 18}{24}h^3 = (\frac{1}{4}\binom{n-1}{3} + 1)h^3$$

i.e. M_{n-3} is a codimension 3 subvariety of $\mathbb{P}(V)$ of degree $\frac{1}{4}\binom{n-1}{3} + 1$.

By a Bertini type theorem, its singular locus is contained in M_{n-5} and it is equal to M_{n-5} if ω is general, for which we have

$$[M_{n-5}] = \det\begin{pmatrix} c_4 & c_5 & c_6 & c_7 \\ c_2 & c_3 & c_4 & c_5 \\ 1 & c_1 & c_2 & c_3 \\ 0 & 0 & 1 & c_1 \end{pmatrix} = c_2c_3c_5 + 2c_1c_4c_5 - c_1c_3c_6 - c_1c_2c_7 - c_5^2 + c_3c_7$$

and in particular it has codimension 10 in $\mathbb{P}(V)$; therefore, M_{n-3} is smooth up to $\mathbb{P}^9$, and is generically singular in dimension 1 in $\mathbb{P}^{11}$. As above, we can make explicit calculations, obtaining

$$[M_{n-5}] = \frac{n(n-1)(n+1)^2(n+2)(n+3)(n^4 - 26n^3 + 311n^2 - 1966n + 5400)}{4838400}h^{10}.$$

4.3. **Equations of the fundamental locus.** Recall from (4.2) that the map

$$\mathbb{P}(f_\omega)\colon \mathbb{P}(V) \to \mathbb{P}(\bigwedge^2 V)^*$$
$$P \mapsto [M_\omega(P)] = \left[\left(\sum_k((-1)^{i+j-1}(a_{i,j,k} - a_{i,k,j} + a_{k,i,j}))x_k(P)\right)_{\substack{i=0,\dots,n\\ j=0,\dots,n}}\right],$$

of (3.9) is the projectivised map on global sections of the bundle map $\phi_\omega\colon \mathcal{T}_{\mathbb{P}(V)}(-1) \to \Omega^1_{\mathbb{P}(V)}(2)$. The degeneracy locus M_r defined at the beginning of Section 4.2 may therefore be interpreted as

$$M_r = \{P \in \mathbb{P}(V) | \operatorname{rank} M_\omega(P) \le r\}.$$

Notation 4.3. Let $F_\omega \subset \mathbb{P}(V)$ denote the locus where the map ϕ_ω drops rank. As a degeneracy locus F_ω has a natural scheme structure. In fact, it is a scheme structure on the fundamental locus of X_ω (see Definition 4.1):

Proposition 4.4. *Let $\omega \in \bigwedge^3 V^*$ be general, and let $M_r \subset \mathbb{P}(V)$ be the degeneracy locus where ϕ_ω has rank at most r. Then $F_\omega = M_{n-2}$ if n is even, and $F_\omega = M_{n-3}$ if n is odd, and is a scheme structure on the fundamental locus of X_ω. The lines of X_ω through a point $P \in M_r \setminus M_{r-2}$, r even, form a star in a $\mathbb{P}^{n-r}$.*

If n is even, then F_ω is a hypersurface of degree $\frac{n}{2} - 1$, it is smooth if $n \le 6$ and it is singular in codimension 5 if $n \ge 8$. If n is odd, then F_ω has codimension 3 and degree $\frac{1}{4}\binom{n-1}{3} + 1$, it is smooth if $n \le 9$ and it is singular in codimension 7 if $n \ge 11$.

Proof. Let $P \in \mathbb{P}(V)$. Without loss of generality we may suppose that $P = [1, 0, \dots, 0] = [e_0]$, and evaluate the matrix M_ω at the point P:

$$M_\omega(P) = \left((-1)^{i+j-1}(a_{0,i,j})\right)_{\substack{i=0,\dots,n\\ j=0,\dots,n}}, \tag{4.5}$$

where we have set $a_{0,j,i} := -a_{0,i,j}$ if $j > i$. Then, if we cut the first row and the first column of $M_\omega(P)$—which are zero—we obtain a submatrix that coincides with the matrix A associated to the homogeneous linear system (3.13) (proof of Proposition 3.7), whose zeros define exactly the lines passing through P.

On one hand, since M_ω defines a map on global sections between bundles of rank n, the rank of $M_\omega(P)$ is at most n. On the other hand, $M_\omega(P)$ is skew symmetric, so it has even rank, so $P \in F_\omega$ if and only if $\mathrm{rank}(M_\omega(P)) \leq n-2$ when n is even, and $\mathrm{rank}(M_\omega(P)) \leq n-3$ when n is odd. Furthermore, if $M_\omega(P)$ has rank r, then there is a linear $\mathbb{P}^{n-r-1} \subset \mathbb{G}$ parametrising lines of the congruence X_ω that pass through P, and vice versa. The dimension, degree formulas for F_ω and its singular locus follow from Section 4.2. □

Remark 4.5. If n is odd, i.e. when the order of X_ω is one, another scheme structure on F_ω is described in [DeP01]. It comes from the interpretation of F_ω as branch locus of the projection from the incidence correspondence to $\mathbb{P}(V)$. With this structure F_ω is non-reduced because its codimension is 3, while from this point of view its expected codimension would be 2.

The fundamental locus F_ω of the congruence X_ω has a natural interpretation in terms of the secant varieties of the Grassmannian. Consider the natural filtration of $\mathbb{P}(\bigwedge^2 V)$ by the secant varieties of the Grassmannian:

$$\mathbb{G} \subset S^1\mathbb{G} \subset \cdots \subset S^{r-1}\mathbb{G} \subset S^r\mathbb{G} = \mathbb{P}(\bigwedge^2 V),$$

where $r = [\frac{n-1}{2}]$. In the dual space $\mathbb{P}(\bigwedge^2 V)^*$ there is a dual filtration. Write $\mathbb{G}' = G(n-1, V^*)$. Then the filtration can be interpreted in the form:

$$\mathbb{G}' \subset S^1\mathbb{G}' \subset \cdots \subset S^r\mathbb{G}' = \mathbb{G}^* \subset \mathbb{P}(\bigwedge^2 V^*).$$

The last secant variety $\mathbb{G}^*$ is the dual of $\mathbb{G}$ parametrising its tangent hyperplanes. If n is odd, the general tangent hyperplane is tangent at one point only (corresponding to a line of $\mathbb{P}(V)$), the previous variety of the filtration parametrises hyperplanes which are tangent along the lines of a 3-space, and so on, until the smallest one corresponds to hyperplanes tangent along the lines of a $\mathbb{P}^{n-2}$. If n is even, the description is similar, but the general tangent hyperplane is tangent along the lines of a 2-plane, and so on.

Corollary 4.6. *Let $\omega \in \bigwedge^3 V^*$ be a 3-form such that $f_\omega \colon V \to \bigwedge^2 V^*$ is injective (condition (GC2)). If we identify $\mathbb{P}(V)$ with $P_\omega = \mathrm{Im}(\mathbb{P}(f_\omega)) \subset \mathbb{P}(\bigwedge^2 V^*)$, i.e. the space of linear equations defining the congruence $X_\omega \subset \mathbb{P}(\bigwedge^2 V)$, then*

$$M_{2k} = P_\omega \cap S^k\mathbb{G}', \qquad\qquad k \geq 0.$$

In particular, the fundamental locus F_ω is the locus of equations whose rank, as a skew-symmetric matrix, is $< n-1$, and, when ω is general, the singular locus of F_ω is the locus of equations whose rank is $< n-3$.

Proof. When f_ω is injective, $P \mapsto [M_\omega(P)]$ defines the identification of $\mathbb{P}(V)$ with P_ω, and so the corollary follows immediately from Proposition 4.4, noting that $S^k\mathbb{G}' \setminus S^{k-1}\mathbb{G}'$ is smooth. □

Finally, we state a theorem illustrating the geometric connection between a congruence of order 1 and its fundamental locus.

Theorem 4.7. *Let ω be a 3-form that satisfies (GC2) on $\mathbb{P}(V)$, with $\dim(V) = n+1$ even. Then X_ω is the closure of the family of $(\frac{n-1}{2})$-secant lines of F_ω.*

Proof. Let $n = 2m+1$ and consider as usual the map $f_\omega \colon V \to \bigwedge^2 V^*$ of (2.5). Up to a change of coordinates, a line L can be written as $L = \langle e_0, e_1 \rangle$. We can interpret the elements of $\bigwedge^2 V^*$ as in (3.10) as $(n+1) \times (n+1)$ skew-symmetric matrices, and therefore, since we are supposing (GC2), the image of L is a pencil of $(n+1) \times (n+1)$ skew-symmetric matrices of the form $M(s,t) = sf_\omega(e_0) + tf_\omega(e_1)$, and each matrix in the pencil has rank at most $n-1$ (see Remark 4.2).

Now, if $[L] \in X_\omega$, then the matrices $M(s,t)$ all contain the same 2-subspace L in their kernel. So on the quotient space by L, the matrices are $(n-1)\times(n-1)$ of generic rank $n-1$. They are all skew-symmetric, so in the pencil there are $(n-1)/2 = m$ matrices of smaller rank.

The same argument can be reversed. If there are m matrices $M(s,t)$ of rank smaller than m, the principal Pfaffians of order m of these matrices have m common zeroes. Therefore their GCD is a non-zero homogeneous polynomial of degree m, and we conclude that they are all proportional. So the matrices $M(s,t)$ must have a common rank 2 subspace L in their kernel. From Lemma 3.2 it follows that $[L] \in X_\omega$. □

Remark 4.8. When F_ω is smooth, the result also follows from [Pes15, Theorem 4.6].

4.4. **Examples.** When $n \le 7$ the natural group action of $\mathrm{SL}(V^*)$ on $\bigwedge^3 V^*$ has finitely many orbits. In particular, there is a unique open orbit, so we list in the examples below, for 3-forms ω of this open orbit, the congruence X_ω and fundamental locus F_ω of X_ω. We start considering the case $n = 3$ in which Lemma 3.2 does not apply.

Example 4.9. If $n = 3$, $\omega \in \bigwedge^3 V^*$ is totally decomposable, so without loss of generality we may assume

$$\omega = x_1 \wedge x_2 \wedge x_3;$$

the equation 3.4 reduces to

$$x_1 p_{23} - x_2 p_{13} + x_3 p_{12} = 0.$$

So $X_\omega = \{p_{12} = p_{13} = p_{23} = 0\} \subset \mathbb{G}$ is the α-plane of lines passing through the point $[e_0] : \{x_1 = x_2 = x_3 = 0\}$, which is F_ω in $\mathbb{P}(V) = \mathbb{P}^3$.

Example 4.10. If $n = 4$, there are two (non-trivial) orbits. If $\omega \in \bigwedge^3 V^*$ belongs to the open orbit it is the product of a 1-form and a general 2-form, so without loss of generality we may assume

$$\omega = x_0 \wedge (x_1 \wedge x_2 + x_3 \wedge x_4).$$

The equation 3.4 reduces to

$$x_0(p_{12} + p_{34}) - x_1 p_{02} + x_2 p_{01} - x_3 p_{04} + x_4 p_{03} = 0.$$

So $X_\omega = \{p_{12} + p_{34} = p_{01} = p_{02} = p_{03} = p_{04} = 0\} \subset \mathbb{G}$ is a smooth quadric threefold, a smooth hyperplane section of the Grassmannian of lines in $\{x_0 = 0\} = \mathbb{P}(V_{x_0}) \subset \mathbb{P}(V)$. In particular, the fundamental locus $F_\omega = \mathbb{P}(V_{x_0})$. As a congruence of $\mathbb{P}^4$ X_ω has order zero and class one, i.e. it is a Schubert variety: $[X_\omega] = \sigma_{2,1}$.

If instead $[\omega]$ lies in the closed orbit, it is totally decomposable, so we can suppose $\omega = x_1 \wedge x_2 \wedge x_3$, and we deduce the equations $p_{12} = p_{13} = p_{23} = 0$, i.e. the condition to be incident to the line $\{x_1 = x_2 = x_3 = 0\}$. In this case, $\dim(X_\omega) = 4$.

Example 4.11. Let $n = 5$ and $\omega \in \bigwedge^3 V^*$. In this case there are 4 orbits. They are described in [Seg17] (see [AOP12] and references therein for modern accounts). In particular, it is shown that the secant variety of $G(3,6)$ is the whole $\mathbb{P}^{19}$, so for ω in the open orbit, we may assume that

$$\omega = x_0 \wedge x_1 \wedge x_2 + x_3 \wedge x_4 \wedge x_5$$

which means $a_{0,1,2} = a_{3,4,5}$ and $a_{i,j,k} = 0$ for $(i,j,k) \neq (0,1,2),(3,4,5)$.

From (3.5) we deduce $p_{0,1} = p_{0,2} = p_{1,2} = p_{3,4} = p_{3,5} = p_{4,5} = 0$, so X_ω is contained in a reducible linear congruence and is given by the lines which meet the two planes $\alpha = \{x_0 = x_1 = x_2 = 0\}$ and $\beta = \{x_3 = x_4 = x_5 = 0\}$ in general position, so $X_\omega = \mathbb{P}^2 \times \mathbb{P}^2$ and $F_\omega = \alpha \cup \beta$.

Since the Schubert cycle which represents the lines meeting a plane is σ_2, by Pieri's formula we have that in the Chow ring of the Grassmannian, our congruence is $[X_\omega] = \sigma_2^2 = \sigma_4 + \sigma_{3,1} + \sigma_{2,2}$, which confirms our calculations in (3.16) that its multidegree is

$(1,1,1)$. We remark also that $\mathrm{rank}(M_\omega) = 4$ for points of $\mathbb{P}^5$ not in the fundamental locus, and $\mathrm{rank}(M_\omega) = 2$ for the points in the fundamental locus, see also Proposition 4.4.

If ω belongs to the second largest orbit, in which case $[\omega]$ is a point on a projective tangent space to $G(3,6)$, then we may assume that

$$\omega = x_0 \wedge x_1 \wedge x_2 + x_2 \wedge x_3 \wedge x_4 + x_4 \wedge x_5 \wedge x_0.$$

Also in this case X_ω has dimension 4, while for ω in the remaining two orbits, the dimension of X_ω is > 4.

Example 4.12. Let $n = 6$ and $\omega \in \bigwedge^3 V^*$. In this case there is an open orbit and 8 other (non-trivial) orbits, see [Sch31] and[AOP12] for explanations and references. If ω belongs to the open orbit, we may assume that

$$\omega = x_1 \wedge x_2 \wedge x_3 + x_4 \wedge x_5 \wedge x_6 + x_0 \wedge (x_1 \wedge x_4 + x_2 \wedge x_5 + x_3 \wedge x_6).$$

The stabiliser of ω is the simple Lie group G_2. The congruence $X_\omega \subset \mathbb{G}$ is the homogeneous variety $\mathrm{G}_2 \subset \mathbb{P}^{13}$, the 5-dimensional closed orbit of the projectivised adjoint representation of G_2, a Fano manifold of index 3. This congruence has order 0, and the fundamental locus is a smooth quadric 5-fold in $\mathbb{P}(V)$ ([FH91], Ch.22, [KR13], [Muk89]). We will say more on this example further on, cf. Example 4.20.

Example 4.13. For $n = 7$, there is an open orbit and 21 other non-trivial orbits ([Gu35], [Gu64], [Oz80], [Đok83], [Ho11]). A representative of the open orbit, according to Ozeki, is

$$\omega = x_0 \wedge x_1 \wedge x_2 + x_0 \wedge x_3 \wedge x_4 + x_1 \wedge x_3 \wedge x_5 + x_1 \wedge x_6 \wedge x_7 + x_2 \wedge x_3 \wedge x_6 + x_2 \wedge x_5 \wedge x_7 + x_4 \wedge x_5 \wedge x_6.$$

Đoković gives a different representative:

$$\omega = x_0 \wedge (x_1 + x_2) \wedge x_3 + x_1 \wedge x_4 \wedge x_5 + x_2 \wedge x_6 \wedge x_7 + x_0 \wedge x_4 \wedge x_6 + x_3 \wedge x_5 \wedge x_7.$$

The variety $X_\omega \subset \mathbb{G}(2, V)$ parametrises the trisecant lines of a general projection in $\mathbb{P}^7$ of the Severi variety $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ ([IM05]). This projection is F_ω. The computation $\deg(X_\omega) = 57$ is [IM05, Proposition 4.6].

Example 4.14. For $n = 8$, there are infinitely many orbits. They are described in [VE88]. It is still possible to write explicitly a general 3-form ω. Indeed, there is a continuous family of semi-simple orbits, depending on 4 parameters, that can be explicitly described. Their union is a Zariski-dense open subset in $\bigwedge^3 V^*$. To write a 3-form ω in this family, we introduce the following notation:

$$\begin{aligned}
p_1 &= x_0 \wedge x_1 \wedge x_2 + x_3 \wedge x_4 \wedge x_5 + x_6 \wedge x_7 \wedge x_8\\
p_2 &= x_0 \wedge x_3 \wedge x_6 + x_1 \wedge x_4 \wedge x_7 + x_2 \wedge x_5 \wedge x_8\\
p_3 &= x_0 \wedge x_4 \wedge x_8 + x_1 \wedge x_5 \wedge x_6 + x_2 \wedge x_3 \wedge x_7\\
p_4 &= x_0 \wedge x_5 \wedge x_7 + x_1 \wedge x_3 \wedge x_8 + x_2 \wedge x_4 \wedge x_6
\end{aligned}$$

Then a general $\omega = \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \lambda_4 p_4$, where the coefficients satisfy $\lambda_1\lambda_2\lambda_3\lambda_4 \neq 0$ and other explicit open conditions. For more details see [VE88].

The moduli space of alternating trilinear forms is related to the moduli space of curves of genus 2, cf. [GS15, GSW13] and references therein. The fundamental locus is a Coble cubic in $\mathbb{P}^8$, whose singular locus is an Abelian surface given as Jacobian of a curve of genus 2 with a $(3,3)-$polarisation. More precisely, the moduli space of alternating 3-forms obtained as GIT quotient $\bigwedge^3 V^* /\!\!/ \mathrm{GL}(V)$ contains a dense subset which is isomorphic to the moduli space of genus 2 curves C with a marked Weierstrass point.

The topological Euler characteristic of the 7-fold X_ω is 0.

Example 4.15. The case $n = 9$ has been studied by Peskine. The congruence X_ω is formed by the 4-secant lines of the fundamental locus F_ω, that is a smooth variety of dimension 6 in $\mathbb{P}^9$, also called *Peskine variety.* F_ω is not quadratically normal. In fact it is on the border of Zak's conjectures on k-normality (cf. [DePM08, Conjecture 1]). X_ω has been considered also in [DV10] in a construction of hyper-Kähler fourfolds.

PROBLEM. Compute the Hodge numbers of X_ω for general ω. The topological Euler characteristic of X_ω for $n = 2t + 2$ for $t = 3, 4, 5, 6$ equals 0, -254, -8412, -284598, so in these cases the derived category of X_ω cannot admit a full exceptional sequence. Is it true that the same thing happens for any $n \geq 8$?

4.5. **Incidence varieties and projective bundles.** Here we look more closely at the relation between a congruence and its fundamental locus. This has been developed also in [Han15, Pes15].

Consider the projective bundle $\mathcal{X} = \mathbb{P}(\mathcal{U}^*)$ over the Grassmannian $\mathbb{G}$. This bundle can be seen as the universal line over $\mathbb{G}$, i.e. the point-line incidence variety in $\mathbb{P}^n \times \mathbb{G}$. Let $\mathcal{O}_\mathcal{X}(\ell)$ be the tautological relatively ample line bundle on $\mathcal{X}$ and

$$\lambda\colon \mathcal{X} \to \mathbb{G}$$

be the projection. It is well-known that there is a canonical isomorphism

$$\mathbb{P}(\Omega^1_{\mathbb{P}(V)}(2)) \simeq \mathcal{X}.$$

Let $\mathcal{O}_\mathcal{X}(h)$ be the relatively ample line bundle on $\mathbb{P}(\Omega^1_{\mathbb{P}(V)}(2))$ and

$$\mu\colon \mathbb{P}(\Omega^1_{\mathbb{P}(V)}(2)) \to \mathbb{P}(V)$$

be the natural projection. Then, we have

$$\lambda^*(\mathcal{O}_\mathbb{G}(1)) \simeq \mathcal{O}_\mathcal{X}(h), \qquad \mu^*(\mathcal{O}_{\mathbb{P}(V)}(1)) \simeq \mathcal{O}_\mathcal{X}(\ell). \tag{4.6}$$

For brevity, we often denote the pull-back of a bundle $\mathcal{E}$ on $\mathbb{P}(V)$ to $\mathcal{X}$ also by $\mathcal{E}$ omitting the symbol μ^*, and likewise for $\mathbb{G}$.

Again we see $\omega \in \bigwedge^3 V^*$ as an element of $H^0(\mathcal{X}, \mathcal{Q}^*(1))$ under the isomorphism:

$$\bigwedge^3 V^* \simeq H^0(\mathbb{G}, \mathcal{Q}^*(1)) \simeq H^0(\mathcal{X}, \mathcal{Q}^*(1)).$$

We let I_ω be the zero locus of ω in this sense, i.e. the zero-locus of the pull-back of φ_ω to $\mathcal{X}$. Clearly $I_\omega \simeq \mathbb{P}(\mathcal{U}^* \mid_{X_\omega})$, i.e. I_ω is the point-line incidence variety restricted to X_ω.

4.5.1. *Locally free resolution of the fundamental locus.* The 3-form ω can be considered as global section of $\Omega^2(3)$ over $\mathbb{P}(V)$, which is to say as the skew-symmetric morphism ϕ_ω of (4.1). Write $\mathcal{C}_\omega$ for the cokernel sheaf of ϕ_ω. Looking back at (4.2) we may write, for even n, the exact sequence:

$$0 \to T_{\mathbb{P}(V)}(-1) \to \Omega^1_{\mathbb{P}(V)}(2) \to \mathcal{C}_\omega \to 0, \tag{4.7}$$

and for odd $n = 2m + 1$, the resolution:

$$0 \to \mathcal{O}_{\mathbb{P}(V)}(1-m) \to T_{\mathbb{P}(V)}(-1) \to \Omega^1_{\mathbb{P}(V)}(2) \to \mathcal{I}_{F_\omega/\mathbb{P}(V)}(m) \to 0, \tag{4.8}$$

where we used $\mathcal{C}_\omega \simeq \mathcal{I}_{F_\omega/\mathbb{P}(V)}(m)$.

Corollary 4.16. *If n is odd and ω satisfies (GC4), then the fundamental locus F_ω is a Fano variety and $\omega_{F_\omega} \simeq \mathcal{O}_{F_\omega}(-3)$.*

Proof. Let $n = 2m+1$. Since ω satisfies (GC4) we have the exact sequence (4.8) and F_ω is a subvariety of $\mathbb{P}(V)$ of pure codimension 3. We have:

$$\mathcal{E}xt^2_{\mathbb{P}(V)}(\mathcal{I}_{F_\omega/\mathbb{P}(V)}, \mathcal{O}_{\mathbb{P}(V)}(-n-1)) \simeq \omega_{F_\omega}.$$

On the other hand, dualising the self-dual exact sequence (4.8), we easily get :

$$\mathcal{E}xt^2_{\mathbb{P}(V)}(\mathcal{I}_{F_\omega/\mathbb{P}(V)}(m), \mathcal{O}_{\mathbb{P}(V)}) \simeq \mathcal{O}_{F_\omega}(m-1).$$

Therefore:

$$\omega_{F_\omega} \simeq \mathcal{O}_{F_\omega}(m-1+m-n-1) \simeq \mathcal{O}_{F_\omega}(-3).$$

□

4.5.2. *Incidence variety and projectivised cokernel sheaf.* The main feature of the cokernel sheaf $\mathcal{C}_\omega$ is that it recovers the tautological $\mathbb{P}^1$-bundle over X_ω, the incidence variety I_ω.

Proposition 4.17. *There is an isomorphism:*

$$I_\omega \simeq \mathbb{P}(\mathcal{C}_\omega).$$

Proof. We noticed that I_ω is the zero-locus in $\mathcal{X}$ of φ_ω, so our goal will be to describe also $\mathbb{P}(\mathcal{C}_\omega)$ this way. So first of all recall the tautological exact sequences:

$$0 \to \mathcal{K} \to \Omega^1_{\mathbb{P}(V)}(2\ell) \to \mathcal{O}_{\mathcal{X}}(h) \to 0,$$
$$0 \to \mathcal{O}_{\mathcal{X}}(h-\ell) \to \mathcal{U}^* \to \mathcal{O}_{\mathcal{X}}(\ell) \to 0,$$

where the vertical tangent bundle $\mathcal{K}$ is defined by the sequence. It is clear that these fit into a commutative exact diagram:

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{K}(h-\ell) & \to & \Omega^1_{\mathbb{P}(V)}(\ell+h) & \to & \mathcal{O}_{\mathcal{X}}(2h-\ell) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{Q}^*(h) & \to & V^* \otimes \mathcal{O}_{\mathcal{X}}(h) & \to & \mathcal{U}^*(h) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_{\mathcal{X}}(h+\ell) & = & \mathcal{O}_{\mathcal{X}}(h+\ell). & &
\end{array}$$

This shows:

$$\mathcal{Q}^*(h) \simeq \mathcal{K}(h-\ell).$$

Remark that, by the sequences (4.7) and (4.8), the variety $\mathbb{P}(\mathcal{C}_\omega)$ is cut in $\mathcal{X} \simeq \mathbb{P}(\Omega^1_{\mathbb{P}(V)}(2))$ by $T_{\mathbb{P}(V)}(-1)$ linearly on the fibres of μ, i.e. it is the zero-locus of a section $\psi_\omega : T_{\mathbb{P}(V)}(-\ell) \to \mathcal{O}_{\mathcal{X}}(h)$. Notice that:

$$\psi_\omega \in H^0(\mathcal{X}, \Omega^1_{\mathbb{P}(V)}(h+\ell)) \simeq H^0(\mathbb{P}(V), \Omega^1_{\mathbb{P}(V)} \otimes \Omega^1_{\mathbb{P}(V)}(3)).$$

Observe that ψ_ω lies in the skew-symmetric part $H^0(\mathbb{P}(V), \Omega^2_{\mathbb{P}(V)}(3))$. In other words, the image of ψ_ω in the summand $H^0(\mathbb{P}(V), S^2\Omega^1_{\mathbb{P}(V)}(3))$ is zero. Now:

$$H^0(\mathbb{P}(V), S^2\Omega^1_{\mathbb{P}(V)}(3)) \simeq H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(2h-\ell)),$$

so ψ_ω goes to zero under the projection $\Omega^1_{\mathbb{P}(V)}(\ell+h) \to \mathcal{O}_{\mathcal{X}}(2h-\ell)$, i.e. it lies in $H^0(\mathcal{X}, \mathcal{K}(h-\ell)) \simeq H^0(\mathcal{X}, \mathcal{Q}^*(h))$. This is compatible with the isomorphism $H^0(\mathbb{P}(V), \Omega^2_{\mathbb{P}(V)}(3)) \simeq \bigwedge^3 V^* \simeq H^0(\mathbb{G}, \mathcal{Q}^*(1))$, so that ψ_ω agrees with φ_ω. □

Let again $M_r \subset \mathbb{P}(V)$ be the locus of points where ϕ_ω has rank at most r. Then Proposition 4.17 and the sequences 4.7 and 4.8 yield

Corollary 4.18. *When n is odd, the incidence variety I_ω is the blow-up of $\mathbb{P}(V)$ along F_ω. When n is even, the restriction of the incidence variety I_ω is a $\mathbb{P}^1$-bundle over $F_\omega \setminus M_{n-4}$, and a $\mathbb{P}^{2k-1}$-bundle over $M_{n-2k} \setminus M_{n-2k-2}$, when $k = 2, ..., (n-2)/2$.*

4.5.3. *Linear sections of the fundamental locus.* This framework can be further used to study linear sections of F_ω and X_ω. Indeed, let $M^* \subset V^*$ and $N^* \subset \bigwedge^2 V^*$ be linear subspaces and we write $X_{\omega,N} = \mathbb{P}(N) \cap X_\omega$ and $I_{\omega,N} = \mathbb{P}(\mathcal{U}^*|_{X_{\omega,N}})$. Also we write $I_{M,\omega}$ for the fibre product of I_ω and $\mathbb{P}(M)$ over $\mathbb{P}(V)$. Assume until the end of the section that $I_{\omega,N}$ and $I_{M,\omega}$ have expected dimension.

Lemma 4.19. *The image in $\mathbb{P}(V)$ of $I_{\omega,N}$ is the degeneracy locus of a map $N \otimes \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{C}_\omega$. Likewise, the image in $\mathbb{G}$ of $I_{M,\omega}$ is the degeneracy locus of a map $M \otimes \mathcal{O}_{X_\omega} \to \mathcal{U}^* |_{X_\omega}$.*

Proof. We treat only $N^* \subset \bigwedge^2 V^*$, the argument for $M^* \subset V^*$ being analogous.

By Proposition 4.17, we have $\mathbb{P}(\mathcal{C}_\omega) \simeq I_\omega$, and we know by (4.6) that $\mathcal{O}_{I_\omega}(h)$ is the pull-back of $\mathcal{O}_{\mathbb{G}}(1)$. So, in I_ω, cutting with $\mathbb{P}(N) \hookrightarrow \mathbb{P}(\bigwedge^2 V)$ corresponds to the vanishing of a morphism $N \otimes \mathcal{O}_{I_\omega} \to \mathcal{O}_{I_\omega}(h)$. This vanishing locus is isomorphic to the projectivised cokernel of the direct image in $\mathbb{P}(V)$ of this map, which again by (4.6) is a morphism $N \otimes \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{C}_\omega$. So the image in $\mathbb{P}(V)$ of $I_{\omega,N}$ is the degeneracy locus of this last morphism. Actually for even n this map factors through $N \otimes \mathcal{O}_{F_\omega} \to \mathcal{C}_\omega$ so the image of $I_{\omega,N}$ is a degeneracy locus inside F_ω. □

Example 4.20. For $n = 4$ and generic ω, F_ω is $\mathbb{P}^3 \subset \mathbb{P}^4$ and $\mathcal{C}_\omega$ is the null correlation bundle on $\mathbb{P}^3$. On the other hand, X_ω is a smooth quadric threefold. The variety I_ω is the projectivisation of the null correlation bundle, which in turn is isomorphic to the projectivised spinor bundle over the quadric threefold. In other words, I_ω is the complete flag for the Lie group Spin(5) or equivalently of Sp(4).

For $n = 6$ and generic ω, the variety I_ω is the complete flag for the exceptional group G_2, F_ω is a smooth 5-dimensional quadric and X_ω is the 5-dimensional homogeneous space $\mathrm{G}_2 / P(\alpha_2)$ of Picard number 1. The sheaf $\mathcal{C}_\omega$ is the rank-2 stable G_2-homogeneous bundle on the quadric F_ω, also called Cayley bundle (cf. [Ott90] for a description in terms of spinor bundles).

Taking a general linear section of codimension 2 of X_ω, i.e. a general subspace $\mathbb{K}^2 = N^* \subset \bigwedge^2 V^*$, one gets a smooth Fano threefold $X_{\omega,N}$ of genus 10. The image in $\mathbb{P}^6 = \mathbb{P}(V)$ of the manifold $\mathbb{P}(\mathcal{U}^*|_{X_{\omega,N}})$ is a determinantal cubic hypersurface of F_ω defined by an injective map of the form:

$$\mathcal{O}_{F_\omega}^2 \to \mathcal{C}_\omega.$$

This is a Fano variety of index 2, singular along a curve of degree 18 and arithmetic genus 10, given by the locus where the map displayed above vanishes. This curve is the image in $\mathbb{P}^6$ of the Hilbert scheme of lines contained in the Fano threefold $X_{\omega,N}$. If L is any such line, $\mathcal{U}^*$ splits over L as $\mathcal{O}_L \oplus \mathcal{O}_L(1)$, so that $\mathcal{O}_L(h)$ contracts $\mathbb{P}(\mathcal{U}^*|_L)$ to a plane with a marked point, which is the point in $\mathbb{P}^6$ that corresponds to the given line L.

4.5.4. *Further degeneracy locus for even n.* Let us briefly study the further degeneracy locus of the bundle map ϕ_ω in case n is even. Set $t = n/2 - 1$. We know that $F_\omega \subset \mathbb{P}^n$ is a hypersurface of degree t, whose singular locus is, generically, $F'_\omega = M_{n-4}$, a subvariety of codimension 6 in $\mathbb{P}^n$.

Recall that the singular locus of F'_ω is, generically, M_{n-6}, a subvariety of codimension 15 in $\mathbb{P}^n$. All these varieties are subcanonical.

To write a locally free resolution of F'_ω, assuming that the codimension is 6 as expected, we use the sheafified Józefiak-Pragacz complex, cf. for instance [Wey03, §6.4.6]. We denote by $\Gamma^{a,b}$ the Schur functor associated with the Young tableau having two columns of sizes a and b. This gives:

$$\begin{aligned} 0 \to \mathcal{O}_{\mathbb{P}^n}(-3t) \to \wedge^2 T_{\mathbb{P}^n}(-2t-3) \to \Gamma^{2t+1,1} T_{\mathbb{P}^n}(-4t-3) \to S^2 T_{\mathbb{P}^n}(-t-3) \oplus \\ \oplus S^2 \Omega_{\mathbb{P}^n}(-2t+3) \to \Gamma^{2t+1,1} \Omega_{\mathbb{P}^n}(t+3) \to \wedge^2 \Omega_{\mathbb{P}^n}(-t+3) \to \mathcal{I}_{F'_\omega/\mathbb{P}^n} \to 0. \end{aligned} \tag{4.9}$$

This resolution is self-dual up to sign and up to twisting by $\mathcal{O}_{\mathbb{P}^n}(-3t)$. In particular, dualising the resolution and using $\omega_{F'_\omega} \simeq \mathcal{E}xt^5_{\mathbb{P}^n}(\mathcal{I}_{F'_\omega/\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(-n-1))$ we get:

Corollary 4.21. *If F'_ω has expected codimension 6 in $\mathbb{P}^n$, then its canonical bundle is*

$$\omega_{F'_\omega} \simeq \mathcal{O}_{F'_\omega}(t-3).$$

For $n \le 6$, F'_ω is just empty. For $n = 8$ i.e. $t = 3$, F'_ω is an Abelian surface, actually the Jacobian of the genus-2 curve associated with ω, embedded by the triple Riemann Theta divisor, appearing as singular locus of the relevant Coble cubic, cf. Example 4.14.

For $n = 10$, i.e. $t = 4$ and generic ω, F'_ω is a smooth canonical 4-fold of degree 99. For $t = 5, 6, 7, 8, 9$ the degree of F'_ω is $364, 1064, 2652, 5871, 11858$.

5. Hilbert scheme

Define the open dense subset $\mathcal{K}$ of $\bigwedge^3 V^*$ by the condition that ω belongs to $\mathcal{K}$ if and only if X_ω has dimension $n-1$. For $\omega \in \mathcal{K}$, we let $P(t)$ be the Hilbert polynomial of X_ω. Then we define $\mathcal{H}$ to be the union of the components of the Hilbert scheme $\mathrm{Hilb}_{P(t)}(\mathbb{P}(\bigwedge^2 V))$ that contain at least one point of the form $[X_\omega]$, with $\omega \in \mathcal{K}$. Sending the proportionality class $[\omega]$ of ω to $[X_\omega]$ we get a morphism:

$$\rho \colon \mathbb{P}(\mathcal{K}) \to \mathcal{H}.$$

Our goal here is to prove the following.

Theorem 5.1. *Let $\dim V = n+1 \ge 6$ and assume that $\omega \in \bigwedge^3 V^*$ satisfies ((GC4)). Then $\mathcal{H}$ is irreducible and smooth at any point $[X_\omega]$ corresponding to $\omega \in \mathcal{K}$. Moreover:*

(i) for $n \ge 6$, ρ embeds $\mathbb{P}(\mathcal{K})$ as an open dense subset of $\mathcal{H}$, so $\dim(\mathcal{H}) = \binom{n+1}{3} - 1$;

(ii) for $n = 5$, ρ is dominant with rational curves as fibres, so $\dim(\mathcal{H}) = \binom{n+1}{3} - 2$.

Proof. Let $n \ge 5$. The algebraic map ρ is defined precisely for all ω lying in $\mathcal{K}$ and the fibres of ρ consist of the forms ω' such that $X_\omega = X_{\omega'}$. Recall that X_ω is determined by its linear span Λ_ω so that $[\omega']$ lies in $\rho^{-1}([X_\omega])$ if and only if $\Lambda_\omega = \Lambda_{\omega'}$. On the other hand Λ_ω is formed by the 2-vectors $L \in \mathbb{P}(\bigwedge^2 V)$ such that $\omega(L) = 0$. Therefore $\rho^{-1}([X_\omega])$ consists of the forms $[\omega']$ such that $\omega'(L) = 0$ for all $L \in \Lambda_\omega$. Since this is a linear condition, we see that the fibre $\rho^{-1}([X_\omega])$ is a linear section of $\mathbb{P}(\mathcal{K})$.

Let now $\omega \in \mathbb{P}(\mathcal{K})$ and recall that X_ω is obtained as zero-locus of the global section φ_ω of $\mathcal{Q}^*(1)$ (see equation (3.14)). Then the normal bundle $\mathcal{N}$ of X_ω in $\mathbb{G}$ is $\mathcal{Q}^*(1)|_{X_\omega}$. Taking the tensor product of $\mathcal{Q}^*(1)$ with the Koszul complex (3.15) we obtain then a resolution over $\mathbb{G}$:

$$\cdots \to \bigwedge^p \mathcal{Q} \otimes \mathcal{Q}^*(1-p) \to \cdots \to \mathcal{Q} \otimes \mathcal{Q}^* \to \mathcal{Q}^*(1) \to \mathcal{N} \to 0,$$

where p ranges from 2 to $n-1$.

It is well-known that $H^0(\mathcal{Q} \otimes \mathcal{Q}^*) \simeq \mathbb{K}$ and $H^0(\mathcal{Q}^*(1)) \simeq \bigwedge^3 V^*$(see Section 3.2). Also it is clear that:

$$H^k(\mathcal{Q} \otimes \mathcal{Q}^*) = H^k(\mathcal{Q}^*(1)) = 0, \qquad \forall k > 0.$$

Taking global sections of the rightmost part of the previous display, since the map $\mathcal{Q} \otimes \mathcal{Q}^* \to \mathcal{Q}^*(1)$ is just φ_ω, we get a linear map:

$$\rho_\omega \colon \bigwedge^3 V^*/\langle \omega \rangle \to H^0(\mathcal{N}),$$

which is nothing but the differential of ρ at $[\omega]$.

Since we already mentioned that the fibres of ρ are linear spaces, we have to check that ρ_ω is an isomorphism for $n \ge 6$, or a surjection with 1-dimensional kernel for $n = 5$. We will also check that, in both cases, $H^k(\mathcal{N}) = 0$ for $k > 0$. This will show that $\mathcal{H}$ is irreducible (as image of $\mathbb{P}(\mathcal{K})$) and smooth over the image of ρ. This will imply that, for $n \ge 6$, $\rho^{-1}[X_\omega]$ is a single point, whereby proving that ρ is a birational morphism, actually an isomorphism over the image of ρ. On the other hand, if $n = 5$, we will deduce that ρ is a surjective morphism whose fibres are of the form $\mathbb{P}^1 \cap \mathbb{P}(\mathcal{K})$, which is a rational (perhaps non compact) curve. Both claims on $\dim(\mathcal{H})$ clearly follow.

In turn, in order to prove the required properties of ρ_ω, for $n \geq 6$ it suffices to check the following vanishing:

$$H^k(\bigwedge^p \mathcal{Q} \otimes \mathcal{Q}^*(1-p)) = 0, \qquad p = 2, \dots, n-2, \qquad \forall k. \tag{5.1}$$

On the other hand, for $n = 5$ we have to verify the same vanishing as (5.1) except for $p = 3$ and $k = 2$, where we have to check:

$$H^2(\bigwedge^3 \mathcal{Q} \otimes \mathcal{Q}^*(-2)) \simeq \mathbb{K}.$$

To prove these facts, we rely on Borel-Bott-Weil's theorem. Indeed, $\bigwedge^p \mathcal{Q} \otimes \mathcal{Q}^*(1-p)$ is an extension of two vector bundles associated with irreducible representations of $\mathrm{GL}(V)$. More precisely, write $\mathcal{E}_\lambda$ for the homogeneous bundle associated with the weight λ of $\mathrm{GL}(V)$ and $\lambda_1, \dots, \lambda_n$ for the fundamental weights. Then, the homogeneous bundles given by the irreducible representations appearing in the filtration of $\bigwedge^p \mathcal{Q} \otimes \mathcal{Q}^*(1-p)$ are $\mathcal{E}_{\lambda_{n+1-p}+\lambda_3-p\lambda_2}$ and $\mathcal{E}_{\lambda_{n+2-p}+(1-p)\lambda_2}$. Now, by Borel-Bott-Weil's theorem, for $n \geq 6$ we have:

$$H^k(\mathcal{E}_{\lambda_{n+1-p}+\lambda_3-p\lambda_2}) = H^k(\mathcal{E}_{\lambda_{n+2-p}+(1-p)\lambda_2}) = 0,$$

for $p = 2, \dots, n-2$, and $\forall k$. This implies (5.1) for $n \geq 6$. On the other hand, for $n = 5$, again Bott's theorem implies the same vanishing except for $p = 3$ and $k = 2$, where we get:

$$H^2(\mathcal{E}_{2\lambda_3-3\lambda_2}) \simeq \mathbb{K}, \qquad H^2(\mathcal{E}_{\lambda_4-2\lambda_2}) = 0.$$

This implies (5.1) for $n = 5$ and thus concludes the proof. □

6. Quadrics containing $\langle X_\omega \rangle$ and congruences in linear subspaces

As usual we denote by Λ_ω the linear span of X_ω. We are interested in characterising the quadrics in the ideal of $\mathbb{G}$ that contain also Λ_ω.

First we compute the dimension of this space of quadrics.

Lemma 6.1. *Let $\omega \in \bigwedge^3 V^*$ be a 3-form such that X_ω has dimension $n-1$, i.e satisfying condition (GC4). Then $h^0(\mathcal{I}_{\Lambda_\omega \cup \mathbb{G}}(2)) = n+1$.*

Proof. The homogeneous ideal of $\Lambda_\omega \cup \mathbb{G}$ is the intersection of the homogeneous ideals of Λ_ω and $\mathbb{G}$, so $H^0(\mathcal{I}_{\Lambda_\omega \cup \mathbb{G}}(2))$ is simply the homogeneous piece of degree 2 of this intersection.

Recall that X_ω is the intersection of $\mathbb{G}$ and Λ_ω, so that we have an equality of homogeneous ideals, $I(X_\omega) = I(\mathbb{G}) + I(\Lambda_\omega)$. Consider therefore the following exact sequence of ideals:

$$0 \to I(\Lambda_\omega) \cap I(\mathbb{G}) \to I(\Lambda_\omega) \oplus I(\mathbb{G}) \to I(X_\omega) \to 0.$$

Looking at the dimension of the homogeneous pieces of degree 2 we get:

$$\dim I(\Lambda_\omega)_2 \cap I(\mathbb{G})_2 = \dim I(\mathbb{G})_2 + \dim I(\Lambda_\omega)_2 - \dim I(X_\omega)_2.$$

Also, we have an exact sequence:

$$0 \to \mathcal{I}_{\mathbb{G}|\mathbb{P}(\bigwedge^2 V)} \to \mathcal{I}_{X_\omega|\mathbb{P}(\bigwedge^2 V)} \to \mathcal{I}_{X_\omega|\mathbb{G}} \to 0.$$

Twisting by $\mathcal{O}_{\mathbb{P}(\bigwedge^2 V)}(2)$ and taking global sections, we easily get:

$$\dim I(X_\omega)_2 = \dim I(\mathbb{G})_2 + h^0(\mathcal{I}_{X_\omega|\mathbb{G}}(2)),$$

and therefore:

$$\dim I(\Lambda_\omega)_2 \cap I(\mathbb{G})_2 = \dim I(\Lambda_\omega)_2 - h^0(\mathcal{I}_{X_\omega|\mathbb{G}}(2)).$$

To finish we have to compute the right-hand-side. On one hand, since $\langle X_\omega \rangle = \Lambda_\omega$ has codimension $n+1$ in $\mathbb{P}(\bigwedge^2 V) = \mathbb{P}^{\binom{n+1}{2}-1}$, we easily get $\dim I(\Lambda_\omega)_2 = n\binom{n+1}{2}$. On the other hand, recall that X_ω is the vanishing locus of the global section φ_ω of $\mathcal{Q}^*(1)$, so that, when the dimension of X_ω is $n-1$, the Koszul complex (3.15) is a resolution of the ideal of X_ω on $\mathbb{G}$. Now we twist this sequence with $\mathcal{O}_{\mathbb{G}}(2)$ and take global sections. Applying Borel-Bott-Weil's theorem, we get:

$$h^0(\mathcal{I}_{X_\omega|\mathbb{G}}(2)) = h^0(\mathcal{Q}(1)) - h^0(\bigwedge^2 \mathcal{Q}) = n\binom{n+1}{2} - n - 1.$$

Putting together, we obtain the equality $\dim I(\Lambda_\omega)_2 \cap I(\mathbb{G})_2 = n+1$. □

We shall find a natural isomorphism

$$V^* \to H^0(\mathbb{P}(\bigwedge^2 V), \mathcal{I}_{\Lambda_\omega \cup \mathbb{G}}(2))$$

parametrising this subspace in the space of quadrics defining $\mathbb{G}$.

Notation 6.2. With notation as in Corollary 2.2, we denote by $\mathcal{Q}_\omega$ the image of the map $V^* \to I(\mathbb{G})$ sending x to $q_{\omega\wedge x}$. These quadratic forms correspond to the quadrics $Q_{\omega\wedge x}$ introduced and studied in Section 2.2.

Proposition 6.3. *Assume that $n \geq 5$ and that X_ω has dimension $n-1$ (condition (GC4)). Then*

$$\mathcal{Q}_\omega = H^0(\mathbb{P}(\bigwedge^2 V), \mathcal{I}_{\Lambda_\omega \cup \mathbb{G}}(2)).$$

Furthermore, for any quadric $Q_{\omega\wedge x} \in \mathcal{Q}_\omega$ of rank $2n$, the linear span $\Lambda_\omega = \langle X_\omega \rangle$ has codimension one in the maximal isotropic subspace Λ^x_ω in $Q_{\omega\wedge x}$.

Proof. The quadrics in $\mathcal{Q}_\omega$ contain Λ_ω in view of Corollary 2.16. Since X_ω has dimension $n-1$, Remarks 3.6 and 2.13 imply that ω is indecomposable. Therefore $V^* \to \mathcal{Q}_\omega$ is an isomorphism, so the first part of the proposition follows. The second part follows from Corollary 2.19. □

We shall denote by X_{ω_x} the congruence in $G(2, V_x)$ defined by the restriction ω_x of ω to V_x. We recall also from 2.9, the subspace Λ_{ω_x}.

Corollary 6.4. *Assume that $n \geq 5$ and that X_ω has dimension $n-1$ (condition (GC4)). If $Q_{\omega\wedge x} \in \mathcal{Q}_\omega$ has rank $2n$, then*

$$\mathrm{Sing}(Q_{\omega\wedge x}) = \Lambda_{\omega_x} = \langle X_{\omega_x} \rangle.$$

Proof. It follows immediately from Proposition 6.3 and Theorem 2.17, (1), because $\langle X_{\omega_x} \rangle = \{[L] \mid x(L) = \omega_x(L) = 0\} = \{[L] \mid x(L) = \omega(L) \wedge x = 0\}$. □

Remark 6.5. If $n = 3$, the only quadric $Q_{\omega\wedge x}$ is the Grassmannian $G(2, V)$, which is smooth, and also X_{ω_x} is empty. If $n = 4$, the rank of $Q_{\omega\wedge x}$ is 6 for any choice of x, so $\mathrm{Sing}(Q_{\omega\wedge x})$ is a $\mathbb{P}^3$. On the other hand $X_{\omega_x} = \Lambda_{\omega_x}$ is a $\mathbb{P}^2$. So the conclusion of Corollary 6.4 is not true for $n = 4$.

Corollary 6.6. *Assume that $n \geq 5$ and that X_ω has dimension $n-1$ (condition (GC4)). If ω_x has rank at least $\frac{n+2}{2}$ for every $x \in V^*$, then there are no 4-forms vanishing on the linear span of X_ω, i.e. on Λ_ω. In particular, there are no quadrics in the ideal of $\mathbb{G}$ that are singular along X_ω.*

Proof. Any 4-form η defines a quadric Q_η in the ideal of $\mathbb{G}$. If η vanishes on Λ_ω, then $X_\omega \subset \mathrm{Sing}(Q_\eta)$. Now, by Proposition 6.3, the linear space Λ_ω is in Q_η only if the 4-form η is of the form $\omega \wedge x$ for some $x \in V^*$. Furthermore, by Lemma 2.7, the quadric $Q_{\omega\wedge x}$ has rank equal to $2 \operatorname{rank} \omega_x$. The space Λ_ω has codimension $n+1$, so it is contained in $\mathrm{Sing}(Q_{\omega\wedge x})$ only if this rank is at most $n+1$, i.e. $2 \operatorname{rank} \omega_x \leq n+1$. □

Note that the assumption on ω_x in Corollary 6.6 is satisfied for general ω. Indeed, let $n = 2m-1$, and let $V = V_0 \oplus V_1$ be a decomposition in m-dimensional subspaces. Let $x \in V_1^*$ and $\omega = \omega_0 + \beta \wedge x$, where $\omega_0 \in \bigwedge^3 V_0^*$ and $\beta \in \bigwedge^2 V_1^*$ are generic forms. Then $\omega_x = \omega_0$ and has rank $m = \frac{n+1}{2}$. The variety X_ω is contained in the singular locus of the quadric $Q_{\omega_0 \wedge x}$ of rank $2m = n+1$.

Similarly, when $n = 2m-2$, let $V = V_0 \oplus V_1$ be a decomposition in a $(m-1)$-dimensional subspace and a m-dimensional subspace. Let $x \in V_1^*$ and $\omega = \omega_0 + \beta \wedge x$, where $\omega_0 \in \bigwedge^3 V_0^*$ and $\beta \in \bigwedge^2 V_1^*$ are generic forms. Then $\omega_x = \omega_0$ and has rank $m-1 = \frac{n}{2}$. The variety X_ω is contained in the singular locus of the quadric $Q_{\omega_0 \wedge x}$ of rank $2m-2 = n$.

The next Theorem gives a more precise formulation of Corollary 6.4.

Theorem 6.7. *Let $n \geq 5$ and assume that $\omega \in \bigwedge^3 V^*$ is a 3-form such that X_ω has dimension $n-1$ (condition (GC4)). Let $x \in V^*$ be a linear form such that ω_x has rank n and $\omega_{x\wedge y}$, the restriction of ω to $V_{x\wedge y}$, has rank $\geq \frac{n+1}{2}$ for every $y \in V_x^*$. Then*

$$\mathrm{Sing}(Q_{\omega\wedge x}) \cap \mathbb{G} = X_{\omega_x}.$$

Conversely, if $n \geq 7$, then $Q_{\omega\wedge x}$ is the unique quadric that contains $\mathbb{G}$ and is singular along X_{ω_x}. Furthermore, the quadric $Q_{\omega\wedge x}$ contains the linear span of $X_{\omega'}$ for any ω' such that $\omega' \wedge x = \omega_x \wedge x$, i.e. $\omega'_x = \omega_x$.

Proof. The quadric $Q_{\omega\wedge x}$ has rank $2n$ because ω_x has rank n by assumption (Lemma 2.7). The first part follows immediately from Corollary 6.4.

For the converse, a general 3-form ω_x on V_x extends to 3-forms on V, which again define a quadric singular on X_{ω_x}. It suffices therefore identify the space of quadrics in the ideal of $\mathbb{G}$ singular along X_{ω_x}.

Again we use the correspondence between 4-forms and quadrics in the ideal of $\mathbb{G}$ (see Corollary 2.2). Let $Q = Q_\eta$ be a quadric singular along $X_{\omega_x} \subset \mathbb{P}(\bigwedge^2 V_x) \subset \mathbb{P}(\bigwedge^2 V)$, where η is a 4-form as above. Write $\eta = \eta_x + \gamma_x \wedge x$ for some 4-form η_x and 3-form γ_x on V_x. Then Q_η is singular along $\langle X_{\omega_x}\rangle$ only if $\eta(L) = 0$ for any $[L] \in \langle X_{\omega_x}\rangle$. But $x(L) = 0$ for any $L \in \bigwedge^2 V_x$, so in this case $\eta(L) = \eta_x(L) + \gamma_x(L) \wedge x = 0$ only if $\eta_x(L) = \gamma_x(L) = 0$. But if $\eta_x(L) = 0$ for any $[L] \in \langle X_{\omega_x}\rangle$, then $\eta_x = 0$ by Corollary 6.6. Hence $\eta = \gamma_x \wedge x$. Furthermore, by Theorem 5.1 the spaces $\{[L] \mid \gamma_x(L) \wedge x = 0\}$ and $\langle X_{\omega_x}\rangle$ are equal if and only if γ_x is proportional to ω_x. Therefore $Q = Q_\eta = Q_{\omega\wedge x} = Q_{\gamma_x \wedge x}$ for any γ_x whose restriction to V_x is proportional to ω_x. In particular, Q depends only on X_{ω_x}, and contains X_ω in a maximal dimensional linear subspace. □

We recall that Theorem 2.17 describes the two families of maximal dimensional subspaces in the quadric $Q_{\omega\wedge x}$ of rank $2n$. One family is in bijection with all 3-forms whose restriction to V_x coincides with ω_x. The congruence X_ω is contained in a unique maximal dimensional linear subspace Λ_ω^x of this family in the quadric $Q_{\omega\wedge x}$. We shall study now the union and the intersection of X_ω and X_{ω_x}.

Theorem 6.8. *Let $\omega \in \bigwedge^3 V^*$ be a 3-form satisfying condition (GC2). Let $x \in V^*$ be general, let V_x be the hyperplane $\{x = 0\}$. Let $e \in V$ be a fixed vector such that $x(e) = 1$. Then let $\omega = \omega_x + \beta_x \wedge x$ be the unique decomposition as in* (1.2). *Moreover, for any $L \in \bigwedge^2 V$ let $L = L_x + e \wedge v_x$ be the unique expression with $L_x \in \bigwedge^2 V_x$ and $v_x \in V_x$. Finally let $Q = Q_{\omega\wedge x}$. Then*

(1) $\Lambda_\omega^x \cap \mathbb{G} = X_\omega \cup X_{\omega_x}$.
(2) If $\Lambda_{\omega'}^x$ is a maximal isotropic space in Q of the same family as Λ_ω^x, for some 3-form ω' such that $\omega'_x = \omega_x$, then $\Lambda_{\omega'}^x \cap \mathbb{G} = X_{\omega'} \cup X_{\omega_x}$.
(3) $X_\omega \cap X_{\omega_x} = X_\omega \cap G(2, V_x)$ is the hyperplane section of X_{ω_x} of equation $\beta_x(L_x) = 0$.

Proof. The inclusion $\supset$ in (1) is clear, because $X_\omega \subset \Lambda_\omega \subset \Lambda_\omega^x$, and $X_{\omega_x} \subset \Lambda_\omega^x$ is contained in the singular locus of Q. Conversely, we have to prove that, if $L \in \bigwedge^2 V$ satisfies the conditions $\omega(L) \wedge x = L \wedge L = 0$, then either $\omega(L) = 0$ or $x(L) = \omega_x(L) = 0$.

Note that $\omega(L) \wedge x = (-\beta_x(v_x) + \omega_x(L_x)) \wedge x = 0$ implies $\omega_x(L_x) = \beta_x(v_x)$. Moreover $x(L) = 0$ if and only if $v_x = 0$, so in this case $[L] \in G(2, V_x)$ and $\omega_x(L_x) = \omega_x(L) = 0$, and we conclude that $L \in X_{\omega_x}$.

If instead $v_x \neq 0$, we fix a basis of V with $e_0 = e$, $e_1 = v_x$, and $x = x_0$, and we write ω and L as follows: $\omega = \omega_{01} + x_0 \wedge \beta_0 + x_1 \wedge \beta_1 + x_0 \wedge x_1 \wedge \alpha$, $L = L_{01} + e_0 \wedge e_1 + e_1 \wedge v_1$, with $\omega_{01} \in \bigwedge^3 \langle x_2, \dots, x_n \rangle$, $\beta_0, \beta_1 \in \bigwedge^2 \langle x_2, \dots, x_n \rangle$, $\alpha \in \langle x_2, \dots, x_n \rangle$, $L_{01} \in \bigwedge^2 \langle e_2, \dots, e_n \rangle$, $v_1 \in \langle e_2, \dots, e_n \rangle$. Now one computes easily that $\omega(L) \wedge x_0 = 0$ is equivalent to $\omega_{01}(L_{01}) - \beta_1(v_1) - \alpha + \beta_1(L_{01})x_1 = 0$, and $\omega(L) = 0$ is equivalent to $\omega(L) \wedge x_0 + x_0\beta_0(L_{01}) = 0$. But $L \wedge L = 0$ is equivalent to $L_{01} \wedge L_{01} - 2e_1 \wedge L_{01} \wedge v_1 + 2e_0 \wedge e_1 \wedge L_{01} = 0$, which implies $L_{01} = 0$. This concludes the proof of (1).

The proof of (2) is similar. To prove (3), note that if $L \in \bigwedge^2 V_x$ then $v_x = 0$, so the condition $\omega(L) \wedge x = 0$ is equivalent to $\omega_x(L_x) = 0$, whereas the condition $\omega(L) = 0$ is equivalent to $\omega_x(L_x) + \beta_x(L_x)x = 0$. Therefore $X_\omega \cap X_{\omega_x}$ is equal to the intersection of X_{ω_x} with the hyperplane of equation $\beta_x(L_x) = 0$.

□

7. The residual congruences and their fundamental loci

As noted in Theorem 3.4, the variety X_ω is contained in a reducible linear congruence, i.e. in a proper section of the Grassmannian by a linear space of codimension $n-1$.

Definition 7.1. Let Γ be a linear subspace of $\mathbb{P}(V)$ of codimension $n-1$, containing X_ω and such that the intersection $Z = \mathbb{G} \cap \Gamma$ is proper. Then the congruence $Y \subset \mathbb{G}$ whose ideal I_Y satisfies $[I_Z \colon I_{X_\omega}] \simeq I_Y$ and $[I_Z \colon I_Y] \simeq I_{X_\omega}$, is called the *residual congruence to* X_ω *in* $\mathbb{G} \cap \Gamma$. We also say that X_ω and Y are *linked in* $\mathbb{G} \cap \Gamma$.

In this case, set-theoretically, we have $\mathbb{G} \cap \Gamma = X_\omega \cup Y$. In Proposition 7.10 we shall give a more precise description of Y in terms of ω and the choice of linear space Γ.

We first describe some general facts about the residual congruence.

Proposition 7.2. *Let Y be the residual congruence to X_ω in $\mathbb{G} \cap \Gamma$.*

(1) For general Γ, Y is an irreducible congruence;
(2) Y has order 1 if n is even and order 0 if n is odd;
(3) Y is locally Cohen–Macaulay.

Proof. (1) follows from Bertini theorem. For (2) it is enough to recall Proposition 3.7 and to note that the linear congruence $X_\omega \cup Y$ has order 1. To prove (3) we recall that the Grassmannian $\mathbb{G} \subset \mathbb{P}(\bigwedge^2 V)$ is arithmetically Gorenstein (aG for short in what follows): in fact, it is subcanonical (the canonical sheaf is $\omega_{\mathbb{G}} \cong \mathcal{O}_{\mathbb{G}}(-n-1)$) and aCM (by Bott's theorem, cf. [Wey03, Theorem 4.1.8]). Therefore, the linear congruence $X_\omega \cup Y$ is—by adjunction—arithmetically Gorenstein as well (see for example [Mig98, Theorem 1.3.3]). But then X_ω and Y are geometrically (directly) G-linked (see [Mig98, Definition 5.1.1]), and the results of [Mig98, Chapter 5] can be applied. In particular, since X_ω is smooth, Y is locally Cohen-Macaulay ([Mig98, Corollary 5.2.13]). □

The linkage will allow us to resolve the ideal of Y in $\mathbb{G}$.

7.1. Resolving the ideal of the residual congruence Y.

Let us recall the Koszul resolution (3.15) of X_ω in $\mathbb{G}$ and write down the Koszul resolution of $X_\omega \cup Y$ in $\mathbb{G}$:

$$0 \to \mathcal{O}_{\mathbb{G}}(1-n) \to \cdots \to \mathcal{O}_{\mathbb{G}}(-p)^{\binom{n-1}{p}} \to \cdots \to \mathcal{O}_{\mathbb{G}} \to \mathcal{O}_{X_\omega \cup Y} \to 0.$$

The surjection $\mathcal{O}_{X_\omega \cup Y} \to \mathcal{O}_{X_\omega}$ lifts to a morphism of complexes that reads:

(7.1)
$$\begin{array}{ccccccccc}
 & & & & & & & & \mathcal{I}_{X_\omega/X_\omega\cup Y} \\
 & & & & & & & & \downarrow \\
\mathcal{O}_{\mathbb{G}}(1-n) & \to \cdots \to & \mathcal{O}_{\mathbb{G}}(-p)^{\binom{n-1}{p}} & \to \cdots \to & \mathcal{O}_{\mathbb{G}} & \longrightarrow & \mathcal{O}_{X_\omega\cup Y} \\
\downarrow & & \downarrow & & \| & & \downarrow \\
\mathcal{O}_{\mathbb{G}}(2-n) & \to \cdots \longrightarrow & \bigwedge^p \mathcal{Q}(-p) & \longrightarrow \cdots \to & \mathcal{O}_{\mathbb{G}} & \longrightarrow & \mathcal{O}_{X_\omega}
\end{array}$$

This yields a resolution of $\mathcal{I}_{X_\omega/X_\omega\cup Y}$ of the form:

$$0 \to \mathcal{O}_{\mathbb{G}}(1-n) \to \cdots \to \mathcal{O}_{\mathbb{G}}(-p)^{\binom{n-1}{p}} \oplus \bigwedge^{p+1} \mathcal{Q}(-p-1) \to \cdots \to \mathcal{Q}(-1) \to \mathcal{I}_{X_\omega/X_\omega\cup Y} \to 0,$$

where $0 \le p \le n-1$. By linkage, taking duals and taking tensor product with $\mathcal{O}_{\mathbb{G}}(1-n)$, we get a resolution of $\mathcal{O}_Y$ and hence of $\mathcal{I}_{Y/\mathbb{G}}$. Using the duality $\bigwedge^p \mathcal{Q}^* \simeq \bigwedge^{n-1-p} \mathcal{Q}(-1)$, this takes the form:

$$(7.2)\quad 0 \to \mathcal{Q}^*(2-n) \to \cdots \to \mathcal{O}_{\mathbb{G}}(-p)^{\binom{n-1}{p}} \oplus \bigwedge^{p-1} \mathcal{Q}(-p) \to \cdots \to \mathcal{O}_{\mathbb{G}}(-1)^n \to \mathcal{I}_{Y/\mathbb{G}} \to 0,$$

with $1 \le p \le n-1$.

Remark 7.3. The map $\mathcal{O}_{\mathbb{G}}(1-n) \to \mathcal{O}_{\mathbb{G}}(2-n)$ in diagram (7.1) defines a Schubert hyperplane containing Y. In fact, by Lemma 2.13, if $\Gamma \supset \Lambda_\omega$ is any codimension $n-1$ linear space of $\mathbb{P}(\bigwedge^2 V)$, then $\Gamma = \Lambda_\omega^{xy}$ (cf. the definition of Λ_ω^{xy} in (2.8)) for a 2-form $x \wedge y \in \bigwedge^2 V^*$. In particular, if Y is residual to X_ω in a linear congruence, then

$$X_\omega \cup Y = \mathbb{G} \cap \Lambda_\omega^{xy}$$

for some $x \wedge y \in \bigwedge^2 V^*$. The Schubert hyperplane defined by $x \wedge y$ intersects Λ_ω^{xy} in $\Lambda_{\omega,x\wedge y}$. We shall show in Proposition 7.5 that $Y = \Lambda_{\omega,x\wedge y} \cap \mathbb{G}$.

Notation 7.4. We denote by $Y_{\omega,x\wedge y}$ the residual congruence to X_ω in $\mathbb{G} \cap \Lambda_\omega^{xy}$, whenever the latter intersection is proper. It will be denoted simply by $Y_{x\wedge y}$ when ω is understood, or Y if also x, y are understood.

The residual congruences to X_ω appear naturally from quadrics of $\mathcal{Q}_\omega$. Let $n \ge 5$ and assume that ω satisfies (GC4). Let us consider the quadric $Q = Q_{\omega\wedge x}$ for some $x \in V^*$. It contains $\mathbb{G}$ and its rank is $2n$, so Q has two $\binom{n}{2}$-dimensional families of maximal dimensional subspaces, i.e. subspaces of codimension n in $\mathbb{P}(\bigwedge^2 V)$. These two families were described in Theorem 2.17. The linear space Λ_ω^x belongs to one of these families, and in view of Proposition 6.3, the linear span $\langle X_\omega \rangle = \Lambda_\omega$ is a hyperplane in Λ_ω^x. A subspace of the other family that intersects Λ_ω^x in codimension one is of the form $\Lambda_{\omega,x\wedge y}$ for some $y \in V^*$. The union $\Lambda_\omega^x \cup \Lambda_{\omega,x\wedge y}$ spans the codimension $n-1$ linear subspace Λ_ω^{xy}. The restriction of $Q_{\omega\wedge x}$ to Λ_ω^{xy} decomposes as

$$Q_{\omega\wedge x} \cap \Lambda_\omega^{xy} = \Lambda_\omega^x \cup \Lambda_{\omega,x\wedge y},$$

and the subspace $\Lambda_{\omega,x\wedge y} \subset \Lambda_\omega^{xy}$ is the intersection of Λ_ω^{xy} with the Schubert hyperplane $\{x \wedge y = 0\} \subset \mathbb{P}(\bigwedge^2 V)$.

Therefore, when $x, y \in V^*$ are general linear forms, the linear congruence $\Lambda_\omega^{xy} \cap \mathbb{G}$ decomposes as

$$\Lambda_\omega^{xy} \cap \mathbb{G} = (\Lambda_\omega^x \cap \mathbb{G}) \cup (\Lambda_{\omega,x\wedge y} \cap \mathbb{G}).$$

By Theorem 6.8 (1), $\Lambda_\omega^x \cap \mathbb{G} = X_\omega \cup X_{\omega_x}$ and, by Theorem 6.7, $\mathrm{Sing}(Q_{\omega\wedge x}) \cap \mathbb{G} = X_{\omega_x}$, so $X_\omega \cup Y_{\omega,x\wedge y} = \Lambda_\omega^{xy} \cap \mathbb{G}$ implies the following lemma:

Proposition 7.5. *The residual congruence $Y_{\omega,x\wedge y}$ is a non-proper linear congruence. More precisely*

$$Y_{\omega,x\wedge y} = \Lambda_{\omega,x\wedge y} \cap \mathbb{G},$$

with codim $\Lambda_{\omega,x\wedge y} = n$. *In particular, the congruence $Y_{\omega,x\wedge y}$ is contained in the Schubert hyperplane formed by the lines meeting $\{x = y = 0\}$. Furthermore,*

$$X_\omega \cap Y_{\omega,x\wedge y}$$

is the intersection of X_ω with this Schubert hyperplane and

$$X_\omega \cap Y_{\omega,x\wedge y} \cong H_{Y_{\omega,x\wedge y}} - X_{\omega_x}$$

as Weil divisors on $Y_{\omega,x\wedge y}$.

Proof. Recalling Lemma 2.16, only the last part remains to be proven. But Λ^x_ω is a hyperplane in Λ^{xy}_ω, so $\Lambda^x_\omega \cap Y_{\omega,x\wedge y} = (X_\omega \cap Y_{\omega,x\wedge y}) \cup X_{\omega_x}$ is a hyperplane section of $Y_{\omega,x\wedge y}$. □

7.2. **The multidegree.** The multidegree of a general congruence $Y = Y_{\omega,x\wedge y}$, residual to X_ω, can be computed for every n using (3.16) and the multidegree of a linear congruence. We recall that in [DeP03, Corollary 2.3] the multidegree $(e_0(n), \dots, e_{[\frac{n-1}{2}]}(n))$ of a linear congruence $B \subset \mathbb{G}$, where

$$e_\ell(n) := \int_{[B]} \sigma_{n-1-\ell,\ell} \qquad 0 \le \ell \le \left[\frac{n-1}{2}\right],$$

is given by the following closed formula

$$\operatorname{multdeg}(B) = \left(1, n-2, \dots, \left(\binom{n-2}{i} - \binom{n-2}{i-2}\right), \dots, \left(\binom{n-2}{\nu} - \binom{n-2}{\nu-2}\right)\right),$$

where $\nu := [\frac{n-1}{2}]$. Similarly (loc. cit.)

$$\deg(B) = \sum_{j=0}^{\nu} \left(\binom{n-2}{j} - \binom{n-2}{j-2}\right)^2 = \frac{1}{n-1}\binom{2n-2}{n}.$$

Anyway, in order to obtain multdeg(Y), it is useful to organise these degrees as we did for those of X_ω in Section 3.2.1. In fact, the proof of Lemma 3.11 applied to $e_i(n) := \left(\binom{n-2}{i} - \binom{n-2}{i-2}\right)$ gives the following lemma:

Lemma 7.6. *The multidegree $(e_i(n))$, $i = 0, \dots, n-1$, satisfies the initial condition*

$$e_0(n) = 1, \qquad n = 2, 3, 4, \dots$$

and the recursion relation

$$e_i(n) = e_{i-1}(n-1) + e_i(n-1)$$

when $i = 1, 2, \dots, [\frac{n-1}{2}]$.

Then, as we did with the multidegree $(d_i(n))$ of X_ω, we may display the multidegree $(e_i(n))$ of B in a triangle with initial entries

$$b_{n,0} = 1, \qquad n = 0, 1, 2, \dots$$

and

$$b_{(i,j)} = b_{(i,j-1)} + b_{(i-1,j)} \qquad i = 1, 2, \dots, \text{ and } j = 1, 2, \dots, i.$$

The multidegree of B is identified as $(e_i(n)) = (b_{(n-1-i,i)})$, $i = 0, \dots, n-1$,

$$(7.3) \qquad (b_{i,j}) = \begin{array}{llllllllll}
1 \\
1 & 1 \\
1 & 2 & 2 \\
1 & 3 & 5 & 5 \\
1 & 4 & 9 & 14 & 14 \\
1 & 5 & 14 & 28 & 42 & 42 \\
1 & 6 & 20 & 48 & 90 & 132 & 132 \\
1 & 7 & 27 & 75 & 165 & 297 & 429 & 429 \\
1 & 8 & 35 & 110 & 275 & 572 & 1001 & 1430 & 1430 \\
1 & 9 & 44 & 154 & 429 & 1001 & 2002 & 3432 & 4862 & 4862
\end{array}$$

for which we have, with the same proof as in Proposition 3.12:

Proposition 7.7. *The degree of a linear congruence $B \subset G(2, n+1)$ is given by the $(n+1)$-st diagonal element*

$$\deg B = b_{(n-1,n-1)} = b_{(n-1,n-2)} = e_n(2n-1)$$

in the above triangle of numbers. The multidegree of B is given by the antidiagonals

$$\operatorname{multdeg}(B) = (e_0(n), e_1(n), \dots, e_{[\frac{n-1}{2}]}(n)) = (b_{(n-1,0)}, b_{(n-2,1)}, \dots, b_{(n-[\frac{n-1}{2}],[\frac{n-1}{2}])}).$$

From this, we can infer the multidegree of Y; in fact, it is enough to "subtract" triangles (3.17) and (7.3) obtaining the following triangle

$$(7.4) \qquad (c_{i,j}) = \begin{array}{llllllllll}
0 \\
1 & 1 \\
0 & 1 & 1 \\
1 & 2 & 3 & 3 \\
0 & 2 & 5 & 8 & 8 \\
1 & 3 & 8 & 16 & 24 & 24 \\
0 & 3 & 11 & 27 & 51 & 75 & 75 \\
1 & 4 & 15 & 42 & 93 & 168 & 243 & 243 \\
0 & 4 & 19 & 61 & 154 & 322 & 565 & 808 & 808 \\
1 & 5 & 24 & 85 & 239 & 561 & 1126 & 1934 & 2742 & 2742
\end{array} ,$$

and if we call $c_{i,j}$ the entries of the triangle (7.4), we have $c_{i,j} = b_{i,j} - a_{i,j}$. Let

$$f_\ell(n) := \int_{[Y]} \sigma_{n-1-\ell,\ell} \qquad\qquad 0 \le \ell \le \left[\frac{n-1}{2}\right],$$

then $(f_0(n), \dots, f_{[\frac{n-1}{2}]}(n))$ is the multidegree of Y. From Propositions 3.12 and 7.7, we obtain the following proposition:

Proposition 7.8. *The degree of $Y = Y_{\omega, x\wedge y} \subset \mathbb{G}$ is the $(n+1)$-st diagonal element*

$$\deg Y = c_{(n-1,n-1)} = c_{(n-1,n-2)} = f_n(2n-1)$$

in the above triangle of numbers. The multidegree of Y is given by the antidiagonals

$$(f_0(2m), f_1(2m), \dots, f_{m-1}(2m)) = (c_{(2m-1,0)}, c_{(2m-2,1)}, \dots, c_{(m,m-1)}),$$

when $n = 2m$, and

$$(f_0(2m-1), f_1(2m-1), \dots, f_{m-1}(2m-1)) = (c_{(2m-2,0)}, c_{(2m-3,1)}, \dots, c_{(m-1,m-1)}),$$

when $n = 2m-1$.

For $3 \le n \le 9$, we get the following multidegree for Y:

$$(0,1), \quad (1,1), \quad (0,2,1), \quad (1,2,3), \quad (0,3,5,3), \quad (1,3,8,8), \quad (0,4,11,16,8).$$

7.3. **Divisors and singularities on the residual congruences to X_ω.** Recall that $Y_{\omega,x\wedge y} = \Lambda_{\omega,x\wedge y} \cap \mathbb{G}$ and thus, by Lemma 2.13, $Y_{\omega,x\wedge y}$ is contained in the pencil of quadrics generated by $Q_{\omega\wedge x}$ and $Q_{\omega\wedge y}$. Furthermore, $Y_{\omega,x\wedge y} = Y_{\omega',x\wedge y'}$ only if $\langle x,y\rangle = \langle x,y'\rangle$, $\omega_x = \omega'_x$ and $\omega_y = \omega'_y$.

Proposition 7.9. *There is an $(n-1)$-dimensional family of congruences $X_{\omega'}$ that are linked to $Y_{\omega,x\wedge y}$ in a linear congruence. They are defined by 3-forms $\omega' = \omega + \alpha' \wedge x \wedge y$, for some $\alpha' \in V^*$.*

To further analyse $Y_{\omega,x\wedge y}$ we consider the pencil of hyperplanes generated by $\mathbb{P}(V_x)$ and $\mathbb{P}(V_y)$:

$$\mathbb{P}(V_{[a:b]}) = \{ax + by = 0\}, \qquad [a:b] \in \mathbb{P}^1$$

and denote by $\omega_{[a:b]}$ the restriction of ω to $V_{[a:b]}$.

Proposition 7.10. *Let $\omega \in \bigwedge^3 V^*$ be a 3-form satisfying condition (GC4). Let $x, y \in V^*$ be linearly independent linear forms, and let $Y_{\omega,x\wedge y}$ be the residual congruence to X_ω in $\mathbb{G} \cap \Lambda^{xy}_\omega$. Then:*

(1) $Y_{\omega,x\wedge y}$ is aCM in its linear span;
(2) X_{ω_x} is a Weil divisor in $Y_{\omega,x\wedge y}$, and $X_{\omega_x} = Y_{\omega,x\wedge y} \cap G(2,V_x)$;
(3) $Y_{\omega,x\wedge y} = \bigcup_{[a:b]\in\mathbb{P}^1} X_{\omega_{[a:b]}}$, where

$$X_{\omega_{[a:b]}} = \{[L] \in G(2,V_{[a:b]}) \subset \mathbb{P}(\bigwedge^2 V_{[a:b]}) \mid \omega_{[a:b]}(L) = 0\} \subset G(2,V_{[a:b]})$$

varies in the pencil of divisors on $Y_{\omega,x\wedge y}$ generated by X_{ω_x} and X_{ω_y}.

Proof. The first statement follows by Gorenstein liaison, see [Mig98, Remark 5.3.2], since $Y_{\omega,x\wedge y}$ is linked to X_ω, which is aCM, see Corollary 3.10, and a linear congruence is aG.

For the second statement note that $\Lambda_{\omega_x} \subset \Lambda_{\omega,x\wedge y}$, so

$$X_{\omega_x} = \Lambda_{\omega_x} \cap G(2,V_x) \subset \Lambda_{\omega,x\wedge y} \cap \mathbb{G} = Y_{\omega,x\wedge y}.$$

But $\dim X_{\omega_x} = \dim Y_{\omega,x\wedge y} - 1$, so (2) follows. Similarly, (2) implies that $X_{\omega_{[a:b]}}$ is a Weil divisor in $Y_{\omega,x\wedge y}$ for any $[a:b] \in \mathbb{P}^1$. Finally, since any $[L] \in Y_{\omega,x\wedge y}$ meets the codimension 2 linear subspace $\mathbb{P}(V_{x\wedge y})$, it lies in $\mathbb{P}(\bigwedge^2 V_{[a:b]})$ for some $[a:b] \in \mathbb{P}^1$, i.e. $[L] \in X_{[a:b]}$. Therefore

$$Y_{\omega,x\wedge y} = \cup_{[a:b]\in\mathbb{P}^1} X_{\omega_{[a:b]}}.$$

□

Notice that in view of Proposition 7.10, $X_{\omega_{[a:b]}} = Y_{\omega,x\wedge y} \cap G(2,V_{[a:b]})$, while $X_{\omega_{[a:b]}} \cap G(2,V_{[a':b']})$ has codimension at least 2 in $X_{\omega_{[a:b]}}$ whenever $[a':b'] \neq [a:b]$. Since every $X_{\omega_{[a:b]}}$ is a divisor in $Y_{\omega,x\wedge y}$, this will allow us to conclude that $Y_{\omega,x\wedge y}$ must be singular.

Lemma 7.11. *For any general $[a':b'] \neq [a:b]$ the intersection $X_{\omega_{[a:b]}} \cap X_{\omega_{[a':b']}}$ has codimension 4 in $Y_{x\wedge y}$.*

Proof. Without loss of generality we can take $X_{\omega_{[a:b]}} = X_{\omega_x}$ and $X_{\omega_{[a':b']}} = X_{\omega_y}$, and we choose coordinates so that $x = x_0$, $y = x_1$. With the usual conventions, we can write

$$\omega = \omega_{01} + \gamma_0 \wedge x_0 + \gamma_1 \wedge x_1 + \alpha_{01} \wedge x_0 \wedge x_1,$$
$$L = L_{01} + w_0 \wedge e_0 + w_1 \wedge e_1 + ce_0 \wedge e_1.$$

With reference to Theorem 6.8 and the notations used in its proof, we get

$$\omega_0 = \omega_{01} + \gamma_1 \wedge x_1, \qquad \beta_0 = \gamma_0 - \alpha_{01} \wedge x_1,$$
$$\omega_1 = \omega_{01} + \gamma_0 \wedge x_0, \qquad \beta_1 = \gamma_1 + \alpha_{01} \wedge x_0,$$
$$L_0 = L_{01} + w_1 \wedge e_1, \qquad v_0 = w_0 - ce_1,$$
$$L_1 = L_{01} + w_0 \wedge e_0, \qquad v_1 = w_1 + ce_0.$$

Consider the intersection

$$X_{\omega_x} \cap X_{\omega_y} = (X_{\omega_x} \cap G(2, V_{x\wedge y}) \cap (X_{\omega_y} \cap G(2, V_{x\wedge y}) \subset X_{\omega_{xy}}.$$

We apply Theorem 6.8 (3) to X_{ω_x} and X_{ω_y}, and get that $X_{\omega_x} \cap G(2, V_{x\wedge y}) = X_{\omega_{01}} \cap H_0$, where H_0 is the hyperplane $\{\gamma_0(w_0) = 0\} \subset \mathbb{P}(\bigwedge^2 V_x)$. So $X_{\omega_x} \cap G(2, V_{x\wedge y})$ has codimension two in X_{ω_x}.

Similarly $X_{\omega_y} \cap G(2, V_{x\wedge y}) = X_{\omega_{01}} \cap H_1$, where H_1 is the hyperplane $\{\gamma_1(w_1) = 0\} \subset \mathbb{P}(\bigwedge^2 V_y)$. So $X_{\omega_y} \cap G(2, V_{x\wedge y})$ has codimension two in X_{ω_y}.

But the hyperplanes H_0, H_1 are distinct if ω is general, so $X_{\omega_x} \cap X_{\omega_y} = X_{\omega_{xy}} \cap H_0 \cap H_1$ has codimension three in X_{ω_x} and hence codimension four in $Y_{x\wedge y}$. □

The hyperplanes H_0, H_1 are equal if, in the expression 3.1 of ω, we have $a_{0,i,j} = a_{1,i,j}$ for any $1 < i < j$. But then $\omega(e_0 - e_1) = 0$, so ω has rank at most n.

Proposition 7.12. *Let $\omega \in \bigwedge^3 V^*$ be a 3-form satisfying condition (GC4). Let $x, y \in V^*$ be general linearly independent linear forms, and let $Y_{\omega,x\wedge y}$ be the residual congruence to X_ω in $\mathbb{G} \cap \Lambda^{xy}_\omega$. The singular locus of the congruence $Y_{\omega,x\wedge y}$ is*

$$\begin{aligned}\mathrm{Sing}(Y_{\omega,x\wedge y}) &= \bigcap_{[a:b]\in\mathbb{P}^1} X_{\omega_{[a:b]}} = \{[L] \in G(2, V_{x\wedge y}) \mid \omega_{x\wedge y}(L) = \omega_x(L) = \omega_y(L) = 0\} \\ &= X_\omega \cap G(2, V_{x\wedge y}) = X_{\omega_{x\wedge y}} \cap H_x \cap H_y,\end{aligned}$$

where $\omega_{x\wedge y}$ is the restriction of ω to $\Pi = \{x = y = 0\}$, and H_x and H_y are the hyperplanes defined—using notation as in Theorem 6.8—by $\beta_x(L_x)$ and $\beta_y(L_y)$, respectively.

In particular the codimension of the singular locus of $Y_{\omega,x\wedge y}$ is 4.

Proof. By Bertini-Kleiman [Kl74] applied to $Q_{\omega\wedge x} \setminus \mathrm{Sing}(Q_{\omega\wedge x})$, $Y_{\omega,x\wedge y}$ is smooth outside $Y_{\omega,x\wedge y} \cap \mathrm{Sing}(Q_{\omega\wedge x}) = Y_{\omega,x\wedge y} \cap \Lambda_{\omega_x} = X_{\omega_x}$ (Proposition 7.10 (2)).

Since the same is true for any linear form $ax + by$, $Y_{\omega,x\wedge y}$ is smooth outside $\cap_{[a:b]\in\mathbb{P}^1} X_{\omega_{[a:b]}}$. But if $y \in \cap_{[a:b]\in\mathbb{P}^1} X_{\omega_{[a:b]}}$, y cannot be a smooth point on Y. If it were, the Weil divisors $X_{\omega_{[a:b]}}$ would be Cartier in a neighbourhood of y, so the intersection of any two of them would have codimension two, contradicting Lemma 7.11.

For the third equality, a computation in coordinates shows that $X_\omega \cap G(2, V_{x\wedge y}) = X_{\omega_x} \cap X_{\omega_y}$. For the fourth we notice that the second equality is equivalent to

$$\mathrm{Sing}(Y_{\omega,x\wedge y}) = X_{\omega_x} \cap X_{\omega_y} \cap X_{\omega_{x\wedge y}};$$

so we conclude by Theorem 6.8, (3),

$$X_{\omega_x} \cap X_{\omega_{x\wedge y}} = X_{\omega_{x\wedge y}} \cap H_x, \qquad X_{\omega_y} \cap X_{\omega_{x\wedge y}} = X_{\omega_{x\wedge y}} \cap H_y.$$

□

Corollary 7.13. *Assume $\omega \in \bigwedge^3 V^*$ satisfies condition (GC4) and let $x, y \in V^*$ be general forms. If $n \leq 4$, then $Y_{\omega,x\wedge y}$ is smooth.*

If $n \geq 5$, $Y_{\omega,x\wedge y}$ is not even factorial in any point $y \in \bigcap_{[a:b]\in\mathbb{P}^1} X_{\omega_{[a:b]}}$. In particular, a general congruence linked to X_ω in a linear congruence is non-singular in codimension 3, but it is not Gorenstein.

The canonical divisor of $Y_{\omega,x\wedge y}$ can be computed.

Proposition 7.14. *Let $n \leq 4$ and let Y be a general residual congruence to X_ω in a linear congruence. Then*

$$K_Y \cong X_{\omega_x} - 3H_Y,$$

as Cartier divisors.

Proof. The union $X_\omega \cup Y$ is a linear congruence with canonical sheaf

$$\omega_{X_\omega \cup Y} \cong \mathcal{O}_{X_\omega \cup Y}(-2).$$

Y is smooth so, by adjunction and Proposition 7.5,

$$K_Y \cong -2H_Y - X_\omega \cap Y \cong -2H_Y - (H_Y - X_{\omega_x}) \cong -3H_Y + X_{\omega_x}.$$

□

7.4. **Fundamental locus of the residual congruence.** Let $Y = Y_{x \wedge y}$ be a residual congruence to X_ω, defined by some general pencil of hyperplanes

$$\mathbb{P}(V_{[a:b]}) = \{ax + by = 0\}, \qquad [a:b] \in \mathbb{P}^1,$$

of $\mathbb{P}(V)$, with base locus $\Pi = \{x = y = 0\}$. Let us denote by G its fundamental locus. Our goal here is to describe G. We distinguish two cases according to whether n is even or odd.

Theorem 7.15. *Assume ω satisfies (GC4) and let Y be the residual congruence to X_ω, associated with a general pencil of hyperplanes of $\mathbb{P}(V)$, with base locus Π. Then the fundamental locus G of Y is*

i) a hypersurface of degree m in $\mathbb{P}(V)$ containing Π and F_ω, when $n = 2m+1$;
ii) the union of Π and of a subvariety G_0 of codimension 3 and degree $(2m^3 - 3m^2 - 5m + 12)/6 = 2\binom{m+1}{3} - \binom{m+1}{2} + 2$, contained in the degree m hypersurface F_ω, when $n = 2m$.

The proof occupies the rest of this section. To start with, let us consider the reducible linear congruence $R = X_\omega \cup Y$ and the universal line $\mathbb{P}(\mathcal{U}^*|_R)$ over R. Following the classical description of degeneracy loci of webs of twisted 2-forms, we write $\mathbb{P}(\mathcal{U}^*|_R)$ as a reducible Palatini scroll, i. e. degeneracy locus of a morphism (cf. for instance [Ott92, FF10]):

$$\mathcal{O}_{\mathbb{P}(V)}^{n-1} \to \Omega_{\mathbb{P}(V)}(2). \tag{7.5}$$

Let $\mathcal{F}$ be the cokernel of this map. We have $\mathbb{P}(\mathcal{F}) \cong \mathbb{P}(\mathcal{U}^*|_R)$. Also, the map (7.5) corresponds to the choice of $n-1$ independent hyperplanes containing X_ω. So it factors through the map $\phi_\omega \colon T_{\mathbb{P}(V)}(-1) \to \Omega_{\mathbb{P}(V)}(2)$ defining F_ω.

For the remaining part of the proof, we distinguish two cases as in the statement according to the parity of n.

7.4.1. *If $n = 2m+1$.* Recall the resolution (4.8) of F_ω. Since the map in (7.5) factors through ϕ_ω, we get an exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_{\mathbb{P}(V)}^{n-1} & = & \mathcal{O}_{\mathbb{P}(V)}^{n-1} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathrm{Im}(\phi_\omega) & \to & \Omega_{\mathbb{P}(V)}(2) & \to & \mathcal{I}_{F_\omega/\mathbb{P}(V)}(m) & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathcal{T} & \to & \mathcal{F} & \to & \mathcal{I}_{F_\omega/\mathbb{P}(V)}(m) & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{7.6}$$

The coherent sheaf $\mathcal{T}$, defined by the diagram, is thus the torsion part of $\mathcal{F}$. Note that the exact sequence on the bottom line, after projectivisation, accounts for the exact sequence

$$0 \to \mathcal{I}_{X_\omega/R} \to \mathcal{O}_R \to \mathcal{O}_{X_\omega} \to 0$$

extracted from (7.1), once taken the universal lines above R and X_ω. Then the fundamental locus G is the support of the sheaf $\mathcal{T}$, which is a hypersurface of degree m, by a straightforward Chern class computation.

To check that G contains Π, we observe that the cokernel of the induced map $\mathcal{O}_{\mathbb{P}(V)}^{n-1} \to T_{\mathbb{P}(V)}(-1)$ is clearly $\mathcal{I}_{\Pi/\mathbb{P}(V)}(1)$. So the leftmost part of the resolution (4.8), combined with the left column of the above diagram, yields another commutative exact diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_{\mathbb{P}(V)}^{n-1} & = & \mathcal{O}_{\mathbb{P}(V)}^{n-1} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_{\mathbb{P}(V)}(1-m) & \to & T_{\mathbb{P}(V)}(-1) & \to & \mathrm{Im}(\phi_\omega) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_{\mathbb{P}(V)}(1-m) & \to & \mathcal{I}_{\Pi/\mathbb{P}(V)}(1) & \to & \mathcal{T} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The bottom row shows that Π lies in the support of $\mathcal{T}$, i.e. G contains Π.

To show that G contains F_ω, we look back at Diagram (7.6). First note that, dualising the middle column, we easily get $\mathcal{E}xt^i_X(\mathcal{F}, \mathcal{O}_X) = 0$ for all $i > 1$. On the other hand, $\mathcal{T}$ is supported on the hypersurface G of degree m. Therefore,

$$\mathcal{E}xt^1_X(\mathcal{T}, \mathcal{O}_X) \simeq \mathcal{H}om_G(\mathcal{T}, \mathcal{O}_G(m))$$

is also supported on G. Also, since F_ω is a Gorenstein subvariety of codimension 3 in $\mathbb{P}(V)$ with $\omega_{F_\omega} \simeq \mathcal{O}_{F_\omega}(-3)$ we have that

$$\mathcal{E}xt^2_X(\mathcal{I}_{F_\omega/\mathbb{P}(V)}(m), \mathcal{O}_X) \simeq \mathcal{O}_{F_\omega}(m-1).$$

Therefore, taking duals of the bottom row of Diagram (7.6) we end up with a surjection:

$$\mathcal{H}om_G(\mathcal{T}, \mathcal{O}_G(m)) \twoheadrightarrow \mathcal{O}_{F_\omega}(m-1).$$

In particular, F_ω is contained in the support of $\mathcal{H}om_G(\mathcal{T}, \mathcal{O}_G(m))$, i.e. in G.

7.4.2. *If* $n = 2m$. We consider the resolution (7.2) of $\mathcal{I}_{Y/\mathbb{G}}$ and twist with $\mathcal{O}_{\mathbb{G}}(1)$. Recalling the setup of §4.5, we lift this resolution to the universal lines over R and Y by pulling back via λ. Finally, we take direct image in $\mathbb{P}(V)$ via μ. The universal line over Y is the projectivisation of $\mu_*(\lambda^*(\mathcal{O}_Y(1))$, cf. [Pes15].

Recall that $\mu_*(\lambda^*(\mathcal{O}_{\mathbb{G}}(1))) \cong \Omega_{\mathbb{P}(V)}(2)$ and that $\mu_*(\lambda^*(\mathcal{O}_{\mathbb{G}})) \cong \mathcal{O}_{\mathbb{P}(V)}$. The remaining terms of the resolution of $\mu_*(\lambda^*(\mathcal{O}_{Y_\omega}(1))$ are computed via Bott's theorem, which provides a long exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}(V)}(1-m) \to \mathcal{O}_{\mathbb{P}(V)}^n \xrightarrow{g} \Omega_{\mathbb{P}(V)}(2) \to \mathcal{I}_{G/\mathbb{P}(V)}(m) \to 0, \tag{7.7}$$

where $\mu_*(\lambda^*(\mathcal{O}_Y(1)) \cong \mathcal{I}_{G/\mathbb{P}(V)}(m)$.

The middle map g in the above sequence is just the result of applying $\mu_* \circ \lambda^*$ to the map $\mathcal{O}_{\mathbb{G}}^n \to \mathcal{O}_{\mathbb{G}}^n(1)$ expressing the generators of $\mathcal{I}_{Y/\mathbb{G}}(1)$. By construction of the resolution of (7.2), the map (7.5) then fits into the long exact sequence above to give the exact commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_{\mathbb{P}(V)}^{n-1} & = & \mathcal{O}_{\mathbb{P}(V)}^{n-1} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathrm{Im}(g) & \to & \Omega_{\mathbb{P}(V)}(2) & \to & \mathcal{I}_{G/\mathbb{P}(V)}(m) & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathcal{T} & \to & \mathcal{F} & \to & \mathcal{I}_{G/\mathbb{P}(V)}(m) & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Again, $\mathcal{T}$ is the torsion part of $\mathcal{F}$, but this time its support is just F_ω. Actually from the leftmost part of (7.7) we can see that $\mathcal{T} \cong \mathcal{O}_{F_\omega}$. Indeed, we have the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_{\mathbb{P}(V)}^{n-1} & = & \mathcal{O}_{\mathbb{P}(V)}^{n-1} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_{\mathbb{P}(V)}(1-m) & \to & \mathcal{O}_{\mathbb{P}(V)}^{n} & \to & \mathrm{Im}(\phi_\omega) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_{\mathbb{P}(V)}(1-m) & \to & \mathcal{O}_{\mathbb{P}(V)} & \to & \mathcal{T} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

and the form of degree $m-1$ appearing in the bottom row must define F_ω. Furthermore, recalling that the map $\mathcal{O}_{\mathbb{P}(V)}^{n-1} \to \Omega_{\mathbb{P}(V)}(2)$ factors through ϕ_ω, i. e. through $T_{\mathbb{P}(V)}(-1)$, we get an exact sequence

$$0 \to \mathcal{I}_{\Pi/\mathbb{P}(V)}(1) \to \mathcal{F} \to \mathcal{C}_\omega \to 0,$$

where $\mathcal{I}_{\Pi/\mathbb{P}(V)}(1)$ again appears as cokernel of $\mathcal{O}_{\mathbb{P}(V)}^{n-1} \to T_{\mathbb{P}(V)}(-1)$.

Also, the torsion part $\mathcal{O}_{F_\omega}$ of $\mathcal{F}$ goes to zero under composition to $\mathcal{I}_{G/\mathbb{P}(V)}(m)$, so we finally get an exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{I}_{\Pi/\mathbb{P}(V)}(1) & = & \mathcal{I}_{\Pi/\mathbb{P}(V)}(1) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_{F_\omega} & \to & \mathcal{F} & \to & \mathcal{I}_{G/\mathbb{P}(V)}(m) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_{F_\omega} & \to & \mathcal{C}_\omega & \to & \mathcal{I}_{\bar{G}/F_\omega}(m) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Here $\bar{G}$ is defined by the bottom row of the diagram as the zero locus of a global section of $\mathcal{C}_\omega$ in F_ω. The subvariety $\bar{G}$ of F_ω has thus codimension 2 in F_ω (and thus codimension 3 in $\mathbb{P}(V)$) by a standard argument relying on the vanishing $H^0(F_\omega, \mathcal{C}_\omega(-1)) = 0$ and on the fact that $\mathrm{Pic}(F_\omega)$ is generated by $\mathcal{O}_{F_\omega}(1)$. A direct Chern class computation shows that $\deg(\bar{G}) = (m+1)(2m^2 - 5m + 6)/6$.

Our goal is to describe the component G_0, the closure of $G \setminus \Pi$. To this end, let us use the rightmost column of the diagram to describe $\bar{G}$ in more detail. The inclusion $\mathcal{I}_{\Pi/\mathbb{P}(V)}(1) \subset \mathcal{I}_{\Pi/\mathbb{P}(V)}(m)$ factors through $\mathcal{I}_{G/\mathbb{P}(V)}(m) \subset \mathcal{I}_{\Pi/\mathbb{P}(V)}(m)$.

On the other hand, if we write

$$G_1 = \Pi \cap F_\omega \tag{7.8}$$

and note that G_1 is a codimension 2 subvariety of F_ω of degree $m-1$, then there is an obvious exact sequence:

$$0 \to \mathcal{I}_{\Pi/\mathbb{P}(V)}(1) \to \mathcal{I}_{\Pi/\mathbb{P}(V)}(m) \to \mathcal{I}_{G_1/F_\omega}(m) \to 0.$$

We are now in position to compute the degree of G_0. Indeed, we have a last exact commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \mathcal{I}_{\Pi/\mathbb{P}(V)}(1) & \to & \mathcal{I}_{G/\mathbb{P}(V)}(m) & \to & \mathcal{I}_{\bar{G}/F_\omega}(m) \to 0 \\
 & & \| & & \downarrow & & \downarrow \\
0 & \to & \mathcal{I}_{\Pi/\mathbb{P}(V)}(1) & \to & \mathcal{I}_{\Pi/\mathbb{P}(V)}(m) & \to & \mathcal{I}_{G_1/F_\omega}(m) \to 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \mathcal{I}_{\Pi/G}(m) & \xrightarrow{\cong} & \mathcal{I}_{G_1/G}(m) \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
\end{array}$$

Since Π and G_0 are irreducible components of G, the ideal $\mathcal{I}_{\Pi/G}$ is supported at G_0 and is torsion-free of rank 1 over G_0. The isomorphism in the lower-right corner induced by the diagram expresses $\bar{G}$ as union of two irreducible components G_0 and G_1, of codimension 2 in F_ω. So the degree of G_0 is computed by $\deg(G_0) = \deg(\bar{G}) - \deg(G_1)$. Plugging in the formula for $\deg(\bar{G})$ and $\deg(G_1) = m-1$, we get the desired expression of $\deg(G_0)$.

□

From this proof we in particular get

Corollary 7.16. *The component G_0 of the fundamental locus of Y is obtained as residual in the zero-locus $\bar{G}$ of a global section of $\mathcal{C}_\omega$ over F_ω with respect to $G_1 = \Pi \cap F_\omega$.*

Finally, we can give a geometric description of the congruence Y as multisecant lines to its fundamental locus, when $n = 2m$, as we did in Theorem 4.7.

Theorem 7.17. *Let $\dim(V) = n+1 = 2m+1$ and assume $\omega \in \bigwedge^3 V^*$ satisfies (GC4) and let Y be the residual congruence to X_ω associated with a general pencil of hyperplanes of $\mathbb{P}(V)$, with base locus Π. Then Y is the closure of the family of $(\frac{n-2}{2})$-secant lines of G_0 that also meet Π, where $G_0 \cup \Pi$ is the fundamental locus of Y.*

Proof. By Proposition 7.10, (3),

$$Y = \bigcup_{[a:b]\in\mathbb{P}^1} X_{\omega_{[a:b]}},$$

where $X_{\omega_{[a:b]}} \subset G(2, V_{[a:b]})$ is defined by $\omega_{[a:b]}(L) = 0$. But $X_{\omega_{[a:b]}}$ is a congruence defined by a 3-form on the n-dimensional vector space $V_{[a:b]}$ where $n = 2m$ is even, so Theorem 4.7 applies. We deduce that the lines of $X_{\omega_{[a:b]}}$ are $(\frac{n-2}{2})$-secants to its fundamental locus: call it $F_{[a:b]}$.

Moreover, by Proposition 7.5, the lines of Y are contained in the Schubert hyperplane of the lines meeting Π. Since Π is a hyperplane in $\mathbb{P}(V_{[a:b]})$, all the lines of $X_{\omega_{[a:b]}}$ meet Π and any line that meets Π is contained in a $\mathbb{P}(V_{[a:b]})$. We infer that

$$\left(\bigcup_{[a:b]\in\mathbb{P}^1} F_{[a:b]}\right) \setminus \Pi = G_0 \setminus \Pi, \qquad \bigcup_{[a:b]\in\mathbb{P}^1} F_{[a:b]} \subset G_0$$

and that any $(\frac{n-2}{2})$-secant line to G_0 that meets Π belongs to one of the $X_{\omega_{[a:b]}}$ and is a line of Y. □

Remark 7.18. In the proof of the preceding theorem, we have proven that—outside Π—G_0 coincides with $\bigcup_{[a:b]\in\mathbb{P}^1} F_{[a:b]}$, where, as in the proof, $F_{[a:b]}$ denotes the fundamental locus of the congruence $X_{\omega_{[a:b]}}$; in other words,

$$G_0 = \overline{\bigcup_{[a:b]\in\mathbb{P}^1} F_{[a:b]}}. \tag{7.9}$$

We can also analyse what happens to G_0 in Π:

Proposition 7.19. *With notations as above, if $n = 2m$, then*

$$G_0 \cap \Pi = G_1 \cap F_{\omega_{x\wedge y}},$$

where $F_{\omega_{x\wedge y}}$ is the fundamental locus of the congruence $X_{\omega_{x\wedge y}}$.

In particular, $G_0 \cap \Pi$ is an improper intersection, it has codimension 2 in Π and degree $(m-1)(m-2)$, and is the complete intersection of $G_1 = F_\omega \cap \Pi$ of degree $m-1$ and $F_{\omega_{x\wedge y}}$ of degree $m-2$.

Proof. Let $P \in G_1 \cap F_{\omega_{x\wedge y}}$; as usual, up to a change of coordinates, we can suppose that $P = [1, 0, \dots, 0] = [e_0]$, we fix a basis $(e_0, \dots, e_n)$ and dual basis $(x_0, \dots, x_n)$ such that $x = x_{n-1}$ and $y = x_n$, i.e. $\Pi^\perp = \langle x_0, \dots, x_{n-2}\rangle$ and we can write uniquely

$$\omega = \omega_{x\wedge y} + \beta_x \wedge x + \beta_y \wedge y + z \wedge x \wedge y,$$

where $\omega_{x\wedge y} \in \bigwedge^3\langle x_0, \dots, x_{n-2}\rangle$, $\beta_x, \beta_y \in \bigwedge^2\langle x_0, \dots, x_{n-2}\rangle$, and $z \in \langle x_0, \dots, x_{n-2}\rangle$.

Since—see Equation (7.8)—$P \in G_1 = F \cap \Pi$, we have that the skew-symmetric matrix M_ω at the point P, defined as in (4.5),

$$M_\omega(P) = \left((-1)^{i+j-1}(a_{0,i,j})\right)_{\substack{i=0,\dots,n\\ j=0,\dots,n}},$$

has rank (at most) $n-2$. The matrix $M_{\omega_{x\wedge y}}$ at the point P is

$$M_{\omega_{x\wedge y}}(P) = \left((-1)^{i+j-1}(a_{0,i,j})\right)_{\substack{i=0,\dots,n-2\\ j=0,\dots,n-2}},$$

i.e. it is the submatrix of $M_\omega(P)$ obtained removing the last 2 rows and 2 columns, and, since $P \in F_{\omega_{x\wedge y}}$, it has rank (at most) $n-4$.

In other words, $P \in G_1 \cap F_{\omega_{x\wedge y}}$ if and only if $\operatorname{rank} M_\omega(P) \le n-2$ and $\operatorname{rank} M_{\omega_{x\wedge y}}(P) \le n-4$, with $\operatorname{rank} M_\omega(P) = \operatorname{rank} M_{\omega_{x\wedge y}}(P) + 2 = n-2$ (if P is general).

It is obvious, from Proposition 4.4, that $\deg G_1 \cap F_{\omega_{x\wedge y}} = (m-1)(m-2)$ and that $G_1 \cap F_{\omega_{x\wedge y}}$ has codimension 2 in Π.

On the other hand, if $P \in G_0 \cap \Pi$, by Proposition 7.10, (3), it belongs to infinitely many lines of the congruence $X_{\omega_{[a:b]}}$ for some $[a:b] \in \mathbb{P}^1$. Up to a change of coordinates, we can suppose that $[a:b] = [1:0]$ and $P = [1, 0, \dots, 0] = [e_0]$, so by Equation (7.9) and Theorem 7.17

$$M_{\omega_x}(P) = \left((-1)^{i+j-1}(a_{0,i,j})\right)_{\substack{i=0,\dots,n-1\\ j=0,\dots,n-1}},$$

i.e. the submatrix of $M_\omega(P)$ without the last row and column, has rank (at most) $n-4$. If follows that $M_\omega(P)$ has rank at most $n-2$ and that $M_{\omega_{x\wedge y}}(P)$ has at most the rank of $M_{\omega_x}(P)$, which is at most $n-4$ therefore $P \in G_1 \cap F_{\omega_{x\wedge y}}$. Therefore, $G_0 \cap \Pi \subset G_1 \cap F_{\omega_{x\wedge y}}$, and they have the same dimension.

To finish the proof, it is sufficient to see that $\deg G_0 \cap \Pi = \deg G_0 - \deg F_{[a:b]} = 2\binom{m+1}{3} - \binom{m+1}{2} + 2 - \left(\frac{1}{4}\binom{2m-2}{3} + 1\right) = (m-1)(m-2)$, by Theorem 7.15 and Proposition 4.4. □

7.5. **Examples.** Let ω be a general 3-form, and Y a general residual congruence to X_ω.

7.5.1. *n=3.* Y is the congruence of lines of a 2-plane G, the plane spanned by the point F_ω and the line Π. The linear congruence $X_\omega \cup Y$ is obtained intersecting $\mathbb{G}$ with a $\mathbb{P}^3$ tangent to $\mathbb{G}$ along a line, which is $X_\omega \cap Y$.

7.5.2. *n=4.* We recall (see Example 4.10) that X_ω is a linear complex of lines in a hyperplane $F_\omega \subset \mathbb{P}^4$. A general Y is obtained by choosing a general 2-plane $\Pi \subset \mathbb{P}^4$, intersecting F_ω is a line G_1. Then G_0 is a line skew to G_1, and Y, which is smooth of degree 3, is the family of lines meeting Π and G_0, i.e. $Y = \mathbb{P}^2 \times \mathbb{P}^1$.

$X_\omega \cap Y$ is a Schubert hyperplane section of a linear complex of lines in $\mathbb{P}^3$, so it is the linear congruence of the lines meeting two skew lines: G_0 and $G_1 = \Pi \cap F_\omega$.

7.5.3. *n=5.* As explained in Example 4.11, X_ω is the congruence of the lines meeting two skew planes α, β. In this case Π is a 3-space contained in $\mathbb{P}(V)$. If we identify $\mathbb{P}(V)$ with $P_\omega \subset \mathbb{P}(\bigwedge^2 V^*)$, it is generated by its intersection with $\mathrm{Sing}(\mathbb{G}^*) \simeq \mathbb{G}$, which is the union of two 2-planes, each one representing the incidence condition to a plane in $\mathbb{P}^5$. The choice of Π identifies a point $A \in \alpha$ and similarly $B \in \beta$, and the line ℓ joining A and B. The residual congruence Y is a fourfold of degree 8 in $\mathbb{P}^9$, of multidegree $(0,2,1)$, representing a family of lines contained in a quadric G. G contains the two planes α, β, Π and all the fundamental loci $F_{[a:b]}$ of the pencil of congruences $X_{\omega_{[a:b]}}$, which are also 3-spaces. Therefore G is a quadric cone of rank 4 with vertex a line. The spaces $F_{[a:b]}$ belong to one family, and Π belongs to the other family. The lines of Y form a subfamily of dimension 4 of the lines contained in one of the two families of 3-spaces contained in G. The singular locus of Y is the point representing the line ℓ, which is also the vertex of G. The intersection $X_\omega \cap Y$ represents the lines meeting α, β and ℓ, so it is a tangent hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^2$, and it results to be the blow up of $\mathbb{P}^3$ at the two points A, B. It is embedded in $\mathbb{P}^7$ with the linear system of quadrics.

7.5.4. *n=6.* X_ω is the G_2-variety, representing a family of lines in a smooth quadric F_ω. Fixed Π, Y is the congruence of lines that are secant to a rational normal scroll G_0 of degree 4 and dimension 3 in $\mathbb{P}^6$ and intersect the 4-space Π. G_0 is smooth because it contains pairs of skew planes, the fundamental loci $F_{[a:b]}$ of the congruences $X_{\omega_{[a:b]}}$. The singular locus of Y is the intersection $X_{\omega_{x\wedge y}} \cap H_0 \cap H_1$, where H_0, H_1 are hyperplanes. But $X_{\omega_{x\wedge y}}$ is a complex of lines in a $\mathbb{P}^3 \subset \Pi$, i.e. a 3-dimensional quadric. We deduce that $\mathrm{Sing}\, Y$ is a conic, the lines of a quadric surface contained in Π. Now consider $G_0 \cap \Pi = G_0 \cap \Pi \cap F_\omega = G_0 \cap G_1$. To understand it, consider first $G_0 \cap \mathbb{P}(V_{[a:b]})$: this is a quartic surface containing the two planes whose union is $F_{[a:b]}$, hence $(G_0 \cap \mathbb{P}(V_{[a:b]}) \setminus F_{[a:b]}$ is a quadric contained in Π. Moreover $G_0 \cap \Pi = G_0 \cap \mathbb{P}(V_x) \cap \mathbb{P}(V_y) \subset \Pi \cap F_\omega = G_1$. Since the missing quadric in $G_0 \cap \mathbb{P}(V_{[a:b]})$ is $G_1 \cap F_{\omega_{x\wedge y}}$, it is the union of the lines of $\mathrm{Sing}\, Y$.

Thus $G_0 \cap \Pi$ is a quadric surface (cf. Proposition 7.19).

7.5.5. *n=7.* The lines of Y are contained in a cubic hypersurface, containing Π and a pencil of projections of $\mathbb{P}^2 \times \mathbb{P}^2$. The singular locus of Y is $(\mathbb{P}^2 \times \mathbb{P}^2) \cap H_x \cap H_y$, a del Pezzo surface of degree 6 in $\mathbb{P}^6$.

7.5.6. *n=8.* For $n = 8$ Y is formed by the trisecant lines of a 5-dimensional variety G_0 of degree 12 that meets $\Pi = \mathbb{P}^6$ in a 4-fold of degree 6. This 4-fold is a complete intersection of the cubic hypersurface $F_\omega \cap \Pi$ and the quadric hypersurface $F_{\omega_{x\wedge y}} \subset \Pi$.

7.5.7. *n=9.* Y represents a family of dimension 8 of lines in a quartic hypersurface containing a $\mathbb{P}^7$ and a pencil of Peskine varieties. $\mathrm{Sing}\, Y$ is formed by the trisecant lines to a del Pezzo surface of degree 6 in $\mathbb{P}^6$.

8. The tables

In tables 1 and 2 we collect some geometrical properties of the congruences we have studied. The notations are as usual:

- ω is a general 3-form in $n+1$ variables;
- $X_\omega \subset \mathbb{G}$ is the congruence of lines where ω vanishes;
- F_ω is the fundamental locus of X_ω;

- Y is the residual congruence of X_ω in a general linear section of $\mathbb{G}$ of codimension $n-1$;
- G is the fundamental locus of Y.

In the last column of table 1 we write the dimension of $\bigwedge^3 V/\operatorname{GL}(n+1)$, the number of moduli of our construction.

n	X_ω	multdeg X_ω	deg X_ω	F_ω	moduli
3	$\mathbb{P}^2$	$(1,0)$	1	$\{*\}$	0
4	$\mathbb{G}(1,3)\cap\mathbb{P}^4$	$(0,1)$	2	$\mathbb{P}^3$	0
5	$\mathbb{P}^2\times\mathbb{P}^2$	$(1,1,1)$	6	$\mathbb{P}^2\cup\mathbb{P}^2$	0
6	G_2	$(0,2,2)$	18	smooth quadric of $\mathbb{P}^6$	0
7	trisecant lines of F_ω	$(1,2,4,2)$	57	general projection of $\mathbb{P}^2\times\mathbb{P}^2$ of degree 6	0
8	7-dimensional family of lines in F_ω	$(0,3,6,6)$	186	Coble cubic hypersurface in $\mathbb{P}^8$ singular along an Abelian surface	3
9	four-secant lines of F_ω	$(1,3,9,12,6)$	622	Peskine variety of degree 15	20

Table 1.

References

[AOP12] H. Abo, G. Ottaviani and C. Peterson, *Non-defectivity of Grassmannians of planes*, J. Algebraic Geom. **21** (2012), no. 1, 1–20.

[AF01] S.I. Agafonov and E.V. Ferapontov, *Systems of conservation laws of Temple class, equations of associativity and linear congruences in P4*, Manuscripta Math. **106**, no. 4 (2001), 461–488.

[BCDeP14] F. Bastianelli, R. Cortini and P. De Poi, *The gonality theorem of Noether for hypersurfaces*, J. Algebraic Geom. **23** no. 2 (2014), 313–339.

[BDeP15] F. Bastianelli, P. De Poi, L. Ein, R. Lazarsfeld and B. Ullery, *Measures of irrationality for hypersurfaces of large degree*, `arXiv:1511.01359`.

[CMR04] C. Ciliberto, M. Mella and F. Russo, *Varieties with one apparent double point*, J. Algebraic Geom. **13**, no. 3 (2004), 475–512.

[CR11] C. Ciliberto and F. Russo, *On the classification of OADP varieties*, Science China Mathematics **54** (2011), 1561–1575.

[DV10] O. Debarre and C. Voisin, *Hyper-Kähler fourfolds and Grassmann geometry*, J. reine angew. Math. **649** (2010), 63–87.

[DeP01] P. De Poi, *On first order congruences of lines of $\mathbb{P}^4$ with a fundamental curve*, Manuscripta Math. **106** (2001), no. 1, 101–116.

[DeP03] ———, *Threefolds in $\mathbb{P}^5$ with one apparent quadruple point*, Comm. Algebra **31** (2003), no. 4, 1927–1947.

[DeP05] ———, *On first order congruences of lines in $\mathbb{P}^4$ with irreducible fundamental surface*, Math. Nachr. **278** (2005), no. 4, 363–378.

[DeP08] ———, *On first order congruences of lines in $\mathbb{P}^4$ with generically non-reduced fundamental surface*, Asian J. Math. **12** (2008), no. 1, 55–64.

[DePM05] P. De Poi and E. Mezzetti, *Linear congruences and systems of conservation laws*, Projective Varieties with Unexpected Properties (Berlin), Walter de Gruyter, KG (2005), pp. 209–230.

[DePM07] ———, *On Congruences of Linear Spaces of Order One*, Rend. Istit. Mat. Univ. Trieste **39** (2007), 177–206.

[DePM08] ———, *Congruences of lines in $\mathbb{P}^5$, quadratic normality, and completely exceptional Monge-Ampère equations*, Geom. Dedicata **131** (2008), 213–230.

[DS01] E. Deutsch and L. Shapiro, *A survey of the Fine numbers*, Discrete Math. **241** (2001), no. 1-3, 241–265.

n	Y	$\operatorname{multdeg} Y$	$\deg Y$	G (= $G_0 \cup \Pi$, n even)	$\Pi \cap G_0$, n even
3	$\mathbb{P}^2$	$(0,1)$	1	$\mathbb{P}^2$	
4	$\mathbb{P}^1 \times \mathbb{P}^2$	$(1,1)$	3	line G_0, plane Π	$\emptyset$
5	4-fold of degree 8 in $\mathbb{P}^9$, with a singular point	$(0,2,1)$	8	quadric of rank 4	
6	secant lines of rational normal 3-fold scroll meeting a $\mathbb{P}^4$	$(1,2,3)$	24	rational normal scroll G_0 of dim 3 and degree 4, $\Pi = \mathbb{P}^4$	quadric surface
7	6-fold of degree 75 singular along a del Pezzo surface of degree 6	$(0,3,5,3)$	75	cubic hypersurface containing a $\mathbb{P}^5$ and $\pi(\mathbb{P}^2 \times \mathbb{P}^2)$	
8	3-secant lines to G_0 that meet a $\mathbb{P}^6$	$(1,3,8,8)$	243	5-fold G_0 of degree 12, $\Pi = \mathbb{P}^6$	complete intersection (2,3) fourfold of degree 6 in $\mathbb{P}^6$
9	8-fold of degree 808 singular along a 4-fold of degree 57	$(0,4,11,16,8)$	808	quartic hypersurface containing a $\mathbb{P}^7$ and a Peskine variety	

TABLE 2.

[Đok83] D. Đoković, *Closures of equivalence classes of trivectors of an eight-dimensional complex vector space*, Canad. Math. Bull. **26** (1) (1983), 92–99.

[FF10] D. Faenzi and M. L. Fania, *Skew-symmetric matrices and Palatini scrolls*, Math. Ann. **347** (2010), no. 4, 859–883.

[Fine] The Online Encyclopedia of Integer Sequences, `https://oeis.org/A000108`.

[FH91] W. Fulton and J. Harris, *Representation theory. A first course*, Springer-Verlag, New York, 1991.

[GH94] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994, Reprint of the 1978 original.

[GS15] L. Gruson and S. V. Sam, *Alternating trilinear forms on a nine-dimensional space and degenerations of* $(3,3)$*-polarized Abelian surfaces*, Proc. Lond. Math. Soc. (3) **110** (2015), no. 3, 755–785.

[GSW13] L. Gruson, S. V. Sam and J. Weyman, *Moduli of Abelian varieties, Vinberg θ-groups, and free resolutions*, Commutative algebra, 419–469, Springer, New York, 2013.

[Gu35] G.B. Gurevich, *Classification of tri-vectors of rank* 8, Dokl. Akad. Nauk. SSSR **2**, (1935), 353–355.

[Gu64] ———, *Foundations of the theory of algebraic invariants*, P. Noordhoff LTD, Groningen (1964).

[Han15] F. Han, *Duality and quadratic normality*, Rend. Istit. Mat. Univ. Trieste **47** (2015), 9–16.

[HT84] J. Harris and L. Tu, *On symmetric and skew-symmetric determinantal varieties*, Topology **23** (1984), no. 1, 71–84.

[Har83] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, New York-Heidelberg-Berlin, 1983, Corr. 3rd printing.

[Ho11] F. Holweck, *Singularities of duals of Grassmannians*, J. Algebra **337** (2011), 369–384.

[IM05] A. Iliev and L. Manivel, *Severi varieties and their varieties of reductions*, J. reine. angew. Math. **585** (2005), 93–139.

[KR13] M. Kapustka and K. Ranestad, *Vector bundles on Fano varieties of genus ten*, Math. Ann. **356** (2013), no. 2, 439–467.

[Kl74] S. L. Kleiman, *The transversality of a general translate*, Compositio Math. **28** (1974), 287–297.

[Lan12] J. M. Landsberg, *Tensors: geometry and applications*, Graduate Studies in Mathematics, 128. American Mathematical Society, Providence, RI, 2012.

[Mig98] J. Migliore, *Introduction to liaison theory and deficiency modules*, Progress in Mathematics, 165. Birkhäuser Boston, Inc., Boston, MA, 1998.

[Muk89] S. Mukai, *Biregular classification of Fano* 3*-folds and Fano manifolds of coindex* 3, Proc. Natl. Acad. Sci. USA **86** (1989), 3000–3002.

[Muk93] ———, *Curves and Grassmannians*, Algebraic geometry and related topics (Inchon, 1992), 19–40, Conf. Proc. Lecture Notes Algebraic Geom., I, Int. Press, Cambridge, MA, 1993.

[Ott90] G. Ottaviani, *On Cayley bundles on the five-dimensional quadric*, Boll. Un. Mat. Ital. A (7) **4** (1990), no. 1, 87–100.

[Ott92] ———, *On 3-folds in* $\mathbb{P}^5$ *which are scrolls*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **19** (1992), no. 3, 451–471.

[Oz80] I. Ozeki, *On the Microlocal Structure of a Regular Prehomogeneous Vector Space Associated with* $GL(8)$, Proc. Japan Acad. **56**, Ser. A (1980), 18–21.

[Pes15] C. Peskine, *Order* 1 *congruences of lines with smooth fundamental scheme*, Rend. Istit. Mat. Univ. Trieste **47** (2015), 203–216.

[Rei07] W. Reichel, *Über die trilinearen alternierenden Formen in* 6 *und* 7 *Veränderlichen*, (Dissertation, Greifswald), (1907).

[Sch31] J. A. Schouten, *Klassifizierung der alternierenden Grössen dritten Grades in* 7 *dimensionen*, Rend. Circ. Mat. Palermo, **55** (1931), 137–156.

[Seg17] C. Segre, *Sui complessi lineari di piani nello spazio a cinque dimensioni*, Ann. Mat. Pura Appl. **27** (1917), 75–123.

[VE88] E.B. Vinberg and A.G. Èlashvili, *Classification of trivectors of a* 9*-dimensional space*, Sel. Math. Sov., **7** (1988), no. 1, 63–98.

[Wey03] J. Weyman, *Cohomology of vector bundles and syzygies*, Cambridge Tracts in Mathematics, 149. Cambridge University Press, Cambridge, 2003.

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